Problems in Classical and Contemporary Mathematics

A curated selection

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Preface

This volume gathers problems the editor has found particularly beautiful over the years, drawn both from the classical canon and from the work of contemporary composers. The selection is curatorial: each problem has been judged on the elegance of its solution, not on originality or difficulty alone, and where a source or composer is known the attribution is recorded. No claim is made to authorship of the problems themselves. The editor's contribution is the selection, the grouping by technique, the typesetting, and the occasional rewriting of solutions that could be tightened.

The chapters are organised by the tool that most naturally unlocks the problems they contain: generating functions for series, parameter differentiation for integrals, conditioning arguments for probability, and so forth. Within each chapter the problems are arranged roughly by ascending technical weight, starting from the accessible and ending in the virtuosic. A diligent reader can work through the book in order; a reader in search of a particular technique can open any chapter and find its emblematic problems at the front.

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Chapter 1

Series and Summations

The problems in this chapter concern closed-form evaluation of infinite series. Generating functions are the recurring instrument: identify an ordinary or exponential generating function whose coefficients match the summand, substitute a convenient value, and read off the answer. The Fibonacci generating function $x/(1-x-x^2)$ and the harmonic generating function $-\ln(1-x)/(1-x)$ each appear here in several disguises. Where the summand splits cleanly mod 4, the Taylor series of sin and cos at $\pm\pi$ unlock unexpected closed forms; this is the engine behind the Raghava sums. Two problems telescope, the quickest method when it applies; one swaps the order of a double sum and reduces to a Beta-function integral. Throughout, the goal is the cleanest path to a closed form, not the shortest. Several problems below were composed by K. S. Raghava and D. Rattaggi and have circulated only in problem columns; the editor records his debt to both.

Problem 1.1

Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{a^{n+1}} = \sum_{n=1}^{\infty} \frac{F_{2n}}{(a+1)^{n+1}},$$

where F_n is the n-th Fibonacci number.

D. Rattaggi

Solution. The ordinary generating function for the Fibonacci sequence is $f(x) = \sum_{n=1}^{\infty} F_n x^n = x/(1-x-x^2)$. The even-indexed subsequence has generating function $g(x^2) = (f(x)+f(-x))/2 = x^2/(1-3x^2+x^4)$, so that $g(x) = \sum_{n=1}^{\infty} F_{2n} x^n = x/(1-3x+x^2)$.

Setting $x = 1/a$ in f:

$$\frac{1}{a} f\left(\tfrac{1}{a}\right) = \sum_{n=1}^{\infty} \frac{F_n}{a^{n+1}} = \frac{1}{a^2-a-1}.$$

Setting $x = 1/(a+1)$ in g:

$$\frac{1}{a+1} g\left(\tfrac{1}{a+1}\right) = \sum_{n=1}^{\infty} \frac{F_{2n}}{(a+1)^{n+1}} = \frac{1}{a^2-a-1}.$$

The two sums agree, as claimed.

Problem 1.2

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lceil n/2 \rceil} - (-1)^n}{n!}.$$

D. Rattaggi

Solution. The parity table of $(-1)^{\lceil n/2 \rceil} - (-1)^n$ has period 4: it takes the values $0, 0, -2, 2$ for $n \equiv 0, 1, 2, 3 \pmod 4$. Therefore

$$S = -2\sum_{n=0}^{\infty} \frac{1}{(4n+2)!} + 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)!}.$$

The quarter-period identities $\sum_n 1/(4n+2)! = \cosh(1) - \cos(1)$ and $\sum_n 1/(4n+3)! = \sinh(1) - \sin(1)$ (each obtained from the four quarter-series of e^x at $x = 1$) give

$$S = \cos(1) - \cosh(1) + \sinh(1) - \sin(1),$$

a value approximately equal to -0.669.

Problem 1.3

Prove that

$$\sum_{n=1}^{\infty} \frac{1+2+\cdots+n}{1\cdot 2\cdots n}(-1)^{1+2+\cdots+n}\,\pi^n \;=\; \frac{\pi(\pi+2)}{2}.$$

K. S. Raghava

Solution. The triangular-number coefficient is $T_n = n(n+1)/2$, and $(-1)^{T_n}$ has period 4 in n: it takes the values $-1, -1, +1, +1$ for $n \equiv 1, 2, 3, 0 \pmod 4$. Write

$$S(x) \;=\; \tfrac{1}{2}\sum_{n\geq 1} \frac{n(n+1)}{n!}\,(-1)^{T_n}\,x^n.$$

Shifting $n \mapsto n+1$ in $n(n+1)x^n/n! = (n+1)x\cdot x^{n-1}/(n-1)! + nx^2\cdot x^{n-2}/(n-2)!$, the sum splits into pieces involving $\sum x^n/(n!)\,(-1)^{T_n}$ and its first derivative. The key identities, obtained by separating the four residue classes mod 4 in the Taylor series of cos and sin, are

$$\sum_{n\geq 0} \frac{x^{2n}}{(2n)!}\,(-1)^n \;=\; \cos x, \qquad \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}\,(-1)^n \;=\; \sin x.$$

Setting $x = \pi$: the cosine series gives $\cos\pi = -1$ and the sine series gives $\sin\pi = 0$. The shifted sums in the expression for $S(\pi)$ collapse to $\pi^2/2$ from the second-order shift and π from the first-order shift, giving

$$S(\pi) \;=\; \frac{\pi^2}{2} + \pi \;=\; \frac{\pi(\pi+2)}{2}.$$

Problem 1.4

Evaluate

$$\sum_{n=0}^{\infty} \arctan\frac{2}{(2n+1)^2}.$$

Solution. Observe the telescoping identity

$$\arctan(2n+2) - \arctan(2n) \;=\; \arctan\frac{2}{(2n+1)^2},$$

obtained by applying $\arctan a - \arctan b = \arctan\big((a-b)/(1+ab)\big)$ to the two adjacent even integers. The original sum therefore telescopes to

$$\sum_{n=0}^{\infty} \arctan\frac{2}{(2n+1)^2} \;=\; \lim_{N\to\infty} \arctan(2N) - \arctan(0) \;=\; \frac{\pi}{2}.$$

PROBLEM 1.5

Evaluate

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution. Differentiating the geometric series $\sum_{n\geq 0} x^n = 1/(1-x)$ with respect to x gives $\sum_{n\geq 1} nx^{n-1} = 1/(1-x)^2$. Therefore $\sum_{n\geq 1} nx^n = x/(1-x)^2$, and setting $x = 1/2$ produces $\sum n/2^n = (1/2)/(1/4) = 2$.

PROBLEM 1.6

Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Solution. Partial fractions give $\frac{1}{n^2-1} = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1})$. The sum telescopes two-at-a-time:

$$\begin{aligned}\sum_{n=2}^{N} \frac{1}{n^2-1} &= \frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\\ &= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}\right).\end{aligned}$$

As $N \to \infty$, the sum approaches $3/4$.

Problem 1.7

Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Solution. This is a classical evaluation due to Euler. The identity

$$\arcsin^2(\tfrac{x}{2}) \;=\; \tfrac{1}{2} \sum_{n\geq 1} \frac{x^{2n}}{n^2\binom{2n}{n}},$$

derived by squaring the Taylor series of arcsin and integrating, yields at $x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} \;=\; 2 \arcsin^2(\tfrac{1}{2}) \;=\; \frac{\pi^2}{18}.$$

Problem 1.8

Evaluate

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the n-th harmonic number.

Solution. Start from the generating function

$$\sum_{n\geq 1} H_n\, x^n \;=\; -\frac{\ln(1-x)}{1-x}.$$

Dividing by x and integrating from 0 to t yields

$$\sum_{n=1}^{\infty} \frac{H_n}{n}\, t^n \;=\; \tfrac{1}{2} \ln^2(1-t) + \mathrm{Li}_2(t),$$

where $\mathrm{Li}_2(t) = \sum t^n/n^2$. Set $t = 1/2$ and substitute the special value $\mathrm{Li}_2(\frac{1}{2}) = \pi^2/12 - (\ln 2)^2/2$:

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n} \;=\; \tfrac{1}{2} \ln^2 2 + \tfrac{\pi^2}{12} - \tfrac{\ln^2 2}{2} \;=\; \frac{\pi^2}{12}.$$

PROBLEM 1.9

Evaluate

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n\cdot k!}.$$

K. S. Raghava

Solution. Swap the order of summation; the inner geometric series in $1/k$ gives

$$S = \sum_{k\geq 2}\frac{1}{k!}\sum_{n\geq 2}\frac{1}{k^n} = \sum_{k\geq 2}\frac{1}{k\,(k-1)\cdot k!}.$$

Use the Beta-function identity $\frac{1}{k(k-1)} = \int_0^1 (1-x)\,x^{k-2}\,dx$ and exchange sum and integral:

$$S = \int_0^1 (1-x)\sum_{k\geq 2}\frac{x^{k-2}}{k!}\,dx = \int_0^1 (1-x)\cdot\frac{e^x-1-x}{x^2}\,dx.$$

Split the integrand:

$$S = \int_0^1 \frac{e^x-1-x}{x^2}\,dx - \int_0^1 \frac{e^x-1-x}{x}\,dx.$$

Integration by parts on the first integral, with $u = e^x-1-x$ and $dv = dx/x^2$, yields $[-(e^x-1-x)/x]_0^1 + \int_0^1 (e^x-1)/x\,dx = -(e-2)+\mathrm{Ein}(1)$. The second integral equals $\mathrm{Ein}(1)-1$ (separating the linear part). Subtracting, the non-elementary $\mathrm{Ein}(1)$ terms cancel:

$$S = (2-e+\mathrm{Ein}(1)) - (\mathrm{Ein}(1)-1) = 3-e.$$

PROBLEM 1.10

Let the sum

$$S = \sum \frac{1}{m^2\,n^2}$$

range over all pairs (m,n) of positive integers such that the largest power of 2 dividing m differs from the largest power of 2 dividing n. Express S in the form $\alpha\,\pi^k$.

R. Stanley

Solution. Write every positive integer as $2^i u$ with u odd. The constraint $i \neq j$ lets the sum factor as

$$S = \left(\sum_{u \text{ odd}} \frac{1}{u^2} \right)^2 \sum_{\substack{i,j \geq 0 \\ i \neq j}} \frac{1}{4^i \cdot 4^j}.$$

The odd-only Basel value is $\sum_{u \text{ odd}} u^{-2} = (1 - 1/4)\,\zeta(2) = \pi^2/8$. The double geometric sum is $(\frac{4}{3})^2 - \frac{16}{15} = 32/45$. Multiplying,

$$S = \frac{\pi^4}{64} \cdot \frac{32}{45} = \frac{\pi^4}{90}.$$

Problem 1.11

Let $p > q$ be the roots of $x^2 - x - 1 = 0$. Define $a_n = (p^n - q^n)/(p - q)$ (the Fibonacci sequence) and let $b_1 = 1$, $b_n = a_{n-1} + a_{n+1}$ for $n \geq 2$. Evaluate

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n}.$$

Solution. The recurrence $a_{n-1} + a_{n+1} = L_n$ (Lucas numbers) means $b_n = L_n$ for $n \geq 2$, with $b_1 = 1$ (slightly smaller than $L_1 = 1$, so no correction needed). Using the closed form $L_n = p^n + q^n$ and summing the two geometric series,

$$\sum_{n \geq 1} \frac{p^n}{10^n} = \frac{p}{10 - p}, \qquad \sum_{n \geq 1} \frac{q^n}{10^n} = \frac{q}{10 - q}.$$

With $p + q = 1$ and $pq = -1$,

$$\sum_{n \geq 1} \frac{L_n}{10^n} = \frac{p(10 - q) + q(10 - p)}{(10 - p)(10 - q)} = \frac{10 - 2pq}{100 - 10 + pq} = \frac{12}{89}.$$

Both 12 and $89 = F_{11}$ emerge from Fibonacci arithmetic.

Chapter 2

Integrals

The problems in this chapter evaluate definite integrals in closed form. Three techniques recur. First, the Feynman parameter trick: introduce a parameter, differentiate under the integral sign, integrate the simpler problem, then recover the original by parameter integration. Second, a substitution that exploits a hidden symmetry; the Weierstrass half-angle substitution converts trigonometric integrals to rational ones, and the involution $x \mapsto 1/x$ folds an integral onto itself. Third, expansion as a series (Taylor or Fourier) followed by termwise integration. One problem evaluates a Riemann sum disguised as a series, and one is computed by recognising the integrand as the expectation of a function of binary digits. The aim throughout is the closed form, not a numerical approximation.

Problem 2.1

Prove that

$$\int_0^{\pi/2} \frac{x}{\tan x}\,dx = \frac{\pi \ln 2}{2}.$$

Solution. Substituting $u = \tan x$ gives

$$\int_0^{\pi/2} \frac{x}{\tan x}\,dx = \int_0^{\infty} \frac{\arctan u}{u\,(u^2+1)}\,du.$$

Parameterise by

$$I(a) = \int_0^{\infty} \frac{\arctan(au)}{u\,(u^2+1)}\,du.$$

Differentiation under the integral sign, using partial fractions on $1/((u^2+1)(a^2u^2+1))$, yields

$$I'(a) \;=\; \frac{\pi}{2(1+a)},$$

so that $I(a) = \frac{1}{2}\pi \ln(1+a)$ after integrating and applying $I(0)=0$. Setting $a=1$ delivers $I(1) = \pi \ln 2/2$.

PROBLEM 2.2

Evaluate

$$\int_0^{\pi/2} \ln(\sin x)\, dx.$$

Solution. Call the integral I. Substituting $x \mapsto \pi/2 - x$ shows $I = \int_0^{\pi/2} \ln(\cos x)\, dx$. Therefore

$$2I \;=\; \int_0^{\pi/2} \ln(\sin x \cos x)\, dx \;=\; \int_0^{\pi/2} \ln\!\left(\frac{\sin 2x}{2}\right) dx.$$

The change of variable $u = 2x$ on the right gives

$$2I \;=\; \frac{1}{2}\int_0^{\pi} \ln(\sin u)\, du \;-\; \frac{\pi}{2}\ln 2.$$

By symmetry the remaining integral is $2I$, so $2I = I - \frac{\pi}{2}\ln 2$, giving $I = -\frac{\pi}{2}\ln 2$.

PROBLEM 2.3

Evaluate

$$\int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}\, F_n}{a^{n+1}} \right) da,$$

where F_n is the n-th Fibonacci number.

K. S. Raghava

Solution. Let $F(x) = \sum_{n\ge 0} F_n x^n = x/(1-x-x^2)$. The generating functions for the even-indexed and odd-indexed Fibonacci subsequences are

$$F_e(x) \;=\; \sum_{n\ge 0} F_{2n} x^n \;=\; \frac{x}{1-3x+x^2},$$

$$F_o(x) = \sum_{n\geq 0} F_{2n+1}x^n = \frac{1-x}{1-3x+x^2}.$$

The sign pattern $(-1)^{n(n-1)/2}$ has period 4 and combines these two subsequences, giving

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}F_n}{a^{n+1}} = \frac{a^2-a+1}{a^4+3a^2+1}.$$

A rational-function integration (partial fractions, logarithmic and arctangent antiderivatives) produces

$$\int_{-\infty}^{\infty} \frac{a^2-a+1}{a^4+3a^2+1}\,da = \frac{2\pi}{\sqrt{5}}.$$

Problem 2.4

Evaluate, for $a > 0$,

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\,dx.$$

Solution. This is a Frullani integral. Writing

$$\frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-tx}\,dt,$$

and swapping the order of integration,

$$\int_0^\infty\int_a^b e^{-tx}\,dt\,dx = \int_a^b\int_0^\infty e^{-tx}\,dx\,dt = \int_a^b \frac{dt}{t} = \ln\frac{b}{a}.$$

Problem 2.5

Evaluate

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Solution. Parameterise: $I(a) = \int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$ for $a \geq 0$. Differentiating under the integral sign:

$$I'(a) = -\int_0^\infty e^{-ax}\sin x\,dx = -\frac{1}{a^2+1}.$$

Integrating from a up to ∞ and using $I(\infty) = 0$,

$$I(a) = \frac{\pi}{2} - \arctan a.$$

Sending $a \to 0^+$ yields $\int_0^\infty \sin x/x\,dx = \pi/2$.

PROBLEM 2.6

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx.$$

Solution. Expanding $\ln(1+x) = \sum_{n\geq 1}(-1)^{n+1}x^n/n$ and integrating term-by-term gives

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12},$$

the alternating Basel sum.

PROBLEM 2.7

Evaluate

$$\int_0^\infty \frac{x}{e^x - 1}\,dx.$$

Solution. Expand the denominator as a geometric series:

$$\frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}.$$

Integrating term-by-term against x:

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \sum_{n=1}^{\infty} \int_0^\infty x\,e^{-nx}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

PROBLEM 2.8

Evaluate

$$\int_0^\infty \frac{1}{1+x^4}\,dx.$$

Solution. Call the integral I. The substitution $x = 1/u$ yields

$$I = \int_0^\infty \frac{u^2}{1+u^4}\,du.$$

Adding this to the original form of I,

$$2I = \int_0^\infty \frac{1+x^2}{1+x^4}\,dx.$$