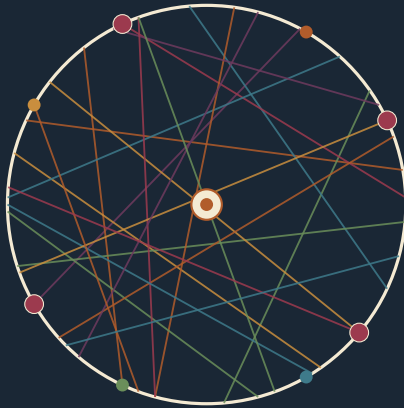




CURATED MATHEMATICS

Geometric Probability



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This volume is a curated gallery of classical results in geometric probability, the corner of mathematics that asks “what is the chance that...?” about shapes drawn at random. The topics themselves are not new; most have been studied since Buffon, Bertrand, or Sylvester. The contribution of this book is a fresh exposition of each one, with full proofs at the level of a first-year undergraduate and figures redrawn at print quality.

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A Note on This Volume



This book is a gallery of geometric probability. Each chapter takes one classical question, the chance that three random sticks form a triangle, that two random chords cross, that four random points in a triangle are in convex position, and gives a full, pictorial proof.

The field has a long and rich history. Georges-Louis Leclerc, Comte de Buffon, posed his *problem of the needle* in 1733: a needle of length ℓ dropped on a floor ruled with parallel lines at spacing $d \geq \ell$ crosses a line with probability $2\ell/(\pi d)$. Joseph Bertrand introduced his *paradox of the random chord* in 1889 to make the point that “random” needs a measure. James Joseph Sylvester posed his *four-point problem* in 1864. J. G. Wendel gave the sign-flip argument that determines the probability n random points on a sphere contain the centre in 1962.

The topics here are not new, and most appear in the standard reference collections of the field, including Maurice Kendall and P. A. P. Moran’s *Geometrical Probability* (Griffin, 1963), Luis Santaló’s *Integral Geometry and Geometric Probability* (Cambridge University Press, 1976), and Daniel A. Klain and Gian-Carlo Rota’s *Introduction to Geometric Probability* (Cambridge University Press, 1997). The contribution of this volume is a fresh exposition of each topic, complete proofs at undergraduate level, and original figures.

The chapters can be read in any order. Each defines its central object from scratch and proves its central result on its own terms. The five parts group the gems thematically, area ratios, symmetry, integration, number-theoretic bridges, measure-

theoretic subtleties, and three-dimensional analogues, but no chapter depends on another.

Every chapter ends with a short Python snippet that verifies the analytic answer by Monte Carlo simulation. Across all 29 chapters the simulated values match the closed forms to within sampling error; we found no discrepancies. The code relies only on `numpy` and can be pasted into an ordinary Python 3 interpreter. Readers are encouraged to modify the snippets, change parameters, vary the sampling rule, try new questions, and so turn the book into a small interactive laboratory.

Contents



Part I

Warmups: area ratios in parameter space

Three sticks forming a triangle



PROBLEM 1.1. Let a, b, c be independent uniform random variables in $[0, 1]$. What is the probability that sticks of lengths a, b, c form a triangle?

Solution. Three positive lengths form a triangle iff each is less than the sum of the other two. The parameter space is the unit cube $[0, 1]^3$, with total volume 1.

The triangle inequality $a < b + c$ *fails* in the region $\{a \geq b + c\}$, which is the corner tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 0, 1)$. Its volume is $\frac{1}{6}$ (one-sixth of the unit cube). By symmetry, the analogous tetrahedra for $b \geq a + c$ and $c \geq a + b$ also have volume $\frac{1}{6}$ each. These three bad regions are disjoint: at most one side can exceed the sum of the other two.

Hence the bad volume is $3 \cdot \frac{1}{6} = \frac{1}{2}$, and the good volume — the triangle region — is $1 - \frac{1}{2} = \frac{1}{2}$. The probability of forming a triangle is

$$\mathbb{P} = \frac{1}{2}.$$



1 Three sticks forming a triangle

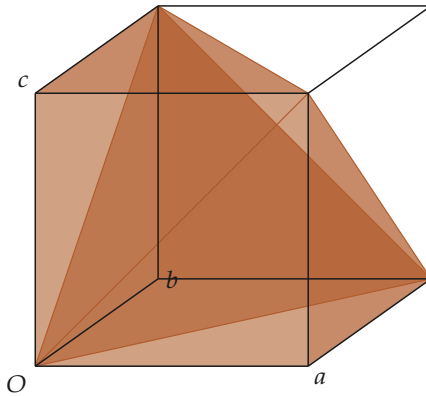


Figure 1.1: The unit cube $[0, 1]^3$ of stick lengths (a, b, c) . The three shaded corner tetrahedra (copper) are the configurations where one stick is at least as long as the other two combined; each has volume $\frac{1}{6}$. The remaining volume $\frac{1}{2}$ is the triangle region.

Monte Carlo verification.

```
import numpy as np
a, b, c = np.random.rand(3, 10**7)
p = ((a+b>c) & (b+c>a) & (c+a>b)).mean()
print(f"sim: {p:.5f}    exact: 0.5")
# sim: 0.50005    exact: 0.5
```

The longest of three pieces



PROBLEM 2.1. A unit stick is snapped at two independent uniform random points. What is the probability that the longest of the three resulting pieces has length greater than $\frac{1}{2}$?

Solution. Let the break points be at positions $x, y \in [0, 1]$, independent uniform. The three pieces have lengths $\min(x, y)$, $|x - y|$, and $1 - \max(x, y)$. The longest piece exceeds $\frac{1}{2}$ iff *some* piece exceeds $\frac{1}{2}$:

$$\begin{aligned} \mathbb{P}(\text{longest} > \tfrac{1}{2}) &= \mathbb{P}(\min(x, y) > \tfrac{1}{2} \text{ or} \\ &\quad |x - y| > \tfrac{1}{2} \text{ or} \\ &\quad 1 - \max(x, y) > \tfrac{1}{2}). \end{aligned}$$

The three events are *mutually exclusive*: if $\min(x, y) > \frac{1}{2}$ then both $x, y > \frac{1}{2}$, forcing $|x - y| < \frac{1}{2}$ and $\max(x, y) > \frac{1}{2}$; similarly if $\max(x, y) < \frac{1}{2}$ then $|x - y| < \frac{1}{2}$. So we may add the three probabilities directly.

Each event corresponds to a region of the unit square:

- $\{\min(x, y) > \frac{1}{2}\} = [\frac{1}{2}, 1]^2$, area $\frac{1}{4}$,
- $\{\max(x, y) < \frac{1}{2}\} = [0, \frac{1}{2}]^2$, area $\frac{1}{4}$,
- $\{|x - y| > \frac{1}{2}\}$ is the union of two corner triangles (upper-left and lower-right), each of area $\frac{1}{8}$, total area $\frac{1}{4}$.

Summing,

$$\boxed{\mathbb{P}(\text{longest} > \tfrac{1}{2}) = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{4}.}$$



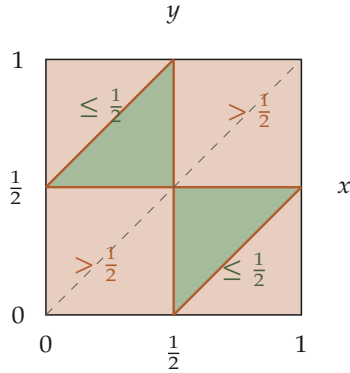


Figure 2.1: The unit square of break points (x, y) . The central sage region — where every piece has length $\leq \frac{1}{2}$ — is a bow-tie of area $\frac{1}{4}$. Its complement (copper), of area $\frac{3}{4}$, is the region where the longest piece exceeds $\frac{1}{2}$.

Monte Carlo verification.

```
import numpy as np
x, y = np.random.rand(2, 10**7)
pieces = np.stack([np.minimum(x,y), np.abs(x-y),
                  0.694120.360780.16864↔ 1-np.maximum(x,y)])
p = (pieces.max(axis=0) > 0.5).mean()
print(f"sim: {p:.5f} exact: 0.75")
# sim: 0.74991 exact: 0.75
```