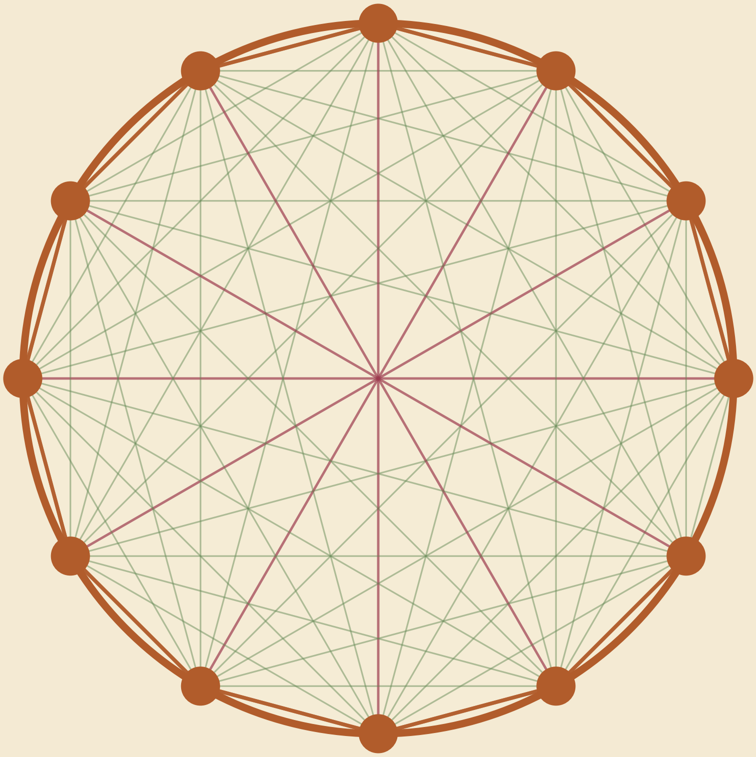
A course in classical algebra



How to See an Equation

The art of algebraic reasoning

Vamshi Jandhyala

London

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Classical algebra through problems

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For every young reader who has ever sat
in front of a quadratic and felt
the floor tilt.

You are not alone. The puzzle is older
than you, and so is the answer.

Mathematics knows no races or geographic boundaries;
for mathematics, the cultural world is one country.

David Hilbert, 1930

That which is sought may yet be restored.

al-Khwārizmī, Baghdad, c. 820

A theorem is a proof a person has been willing to finish.

on the first page of a working notebook

A Note to the Reader

This book is for the reader who wants to *do* algebra, not merely read it. It is written for the ambitious and motivated student who has met the usual school syllabus and wants to meet what lies beyond.

Its purpose is twofold. First, to teach rigorous mathematical reasoning: how to read a problem carefully, how to choose a technique, how to write a proof that someone else can follow. Second, to display the beautiful and powerful techniques that classical algebra has accumulated: the Euclidean algorithm, Viète's formulas, the factor theorem, Newton's identities, the classical inequalities, and the rest. Every problem in the book is a small piece of that inheritance.

Each chapter is a carefully chosen set of problems, grouped into short movements by technique. Almost every problem reveals a distinct idea, and the techniques rarely repeat; the few foundational results that recur, such as the arithmetic-geometric mean inequality, return as reusable tools rather than as repeated exercises. The problems are classical, picked from the problem-solving tradition that has stood up over decades of use, and they favour concrete questions ("find the value of", "solve for x", "compute the remainder when …") over abstract theorems.

Every problem is followed by a discussion. The discussion names the central technique, gives the shortest honest proof or computation, works the steps through in full, and marks the trap that catches the unwary.

Part I, *Foundations*, assembles the instruments: proof, number systems, algebraic identities, functions, inequalities, induction, and exponentials and logarithms. Part II, *Algebra*, puts them to work: quadratic and polynomial equations, complex numbers, sequences, counting, the binomial theorem, the classical inequalities, and functional equations. Each chapter requires only those that precede it; each exists to unlock the next.

The chapters are best read in order. A reader who needs a specific tool, however, should feel free to jump: the dependency graph is explicit, and every definition is restated where it first bears weight.

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Part I

Foundations

CHAPTER 1

Proof and logic

DIRECT PROOF AND CONTRAPOSITIVE

PROBLEM 1.1. Let n be any integer. Show that $n^5 - n$ is divisible by 30.

Technique. Direct proof: work forward from the hypothesis using algebra and a little case-checking.

Discussion. Since $30 = 2 \cdot 3 \cdot 5$, and $2, 3, 5$ have no common factor among themselves, it is enough to show that $n^5 - n$ is divisible by each of $2, 3$, and 5 separately. (If an integer is divisible by three numbers that share no common factor, it must be divisible by their product.)

Factor:

$$n^5 - n \;=\; n(n^2-1)(n^2+1) \;=\; (n-1)\,n\,(n+1)(n^2+1).$$

Divisible by 2 *and by* 3. Among any two consecutive integers, one is even; among any three, one is a multiple of 3. So the first three factors $(n-1) \cdot n \cdot (n+1)$ already contribute both a factor of 2 and a factor of 3.

Divisible by 5. We check the five possible remainders when n is divided by 5.

- If n leaves remainder 0: then n itself is divisible by 5.
- If n leaves remainder 1: then $n - 1$ is divisible by 5.
- If n leaves remainder 4: then $n + 1$ is divisible by 5.

- If n leaves remainder 2: then n^2 leaves remainder 4, so n^2+1 leaves remainder 5, which is 0. So n^2+1 is divisible by 5.
- If n leaves remainder 3: similarly, n^2 leaves remainder 9, which is 4, so n^2+1 is again divisible by 5.

In every case, at least one of the four factors of n^5-n is divisible by 5.

Combining: n^5-n is divisible by $2, 3$, and 5, and therefore by 30. □

What to take away. A "divisible by 30" target breaks up into "divisible by 2", "divisible by 3", and "divisible by 5". Each piece is attacked separately, and then the pieces are multiplied back together. The clever step here is the factoring $n^5-n = (n-1)\cdot n\cdot(n+1)\cdot(n^2+1)$; without it, the problem is much harder.

Problem 1.2. Let $n \geq 2$ be an integer. Show that if 2^n-1 is prime, then n itself must be prime.

Technique. Contrapositive: the statement "if P, then Q" says the same thing as "if not Q, then not P." Sometimes the second form is easier to prove.

Discussion. The original statement is: if 2^n-1 is prime, then n is prime. The contrapositive is: if n is *not* prime, then 2^n-1 is *not* prime. We prove the contrapositive.

Suppose n is not prime. Then $n = ab$ for some integers a, b with $1 < a, b < n$. We use a useful algebraic identity (essentially the sum of a geometric progression): for any number x,

$$x^b - 1 = (x-1)(x^{b-1} + x^{b-2} + \cdots + x + 1).$$

Set $x = 2^a$. Then $x^b = 2^{ab} = 2^n$, and the identity becomes

$$2^n - 1 = (2^a-1)\cdot(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1).$$

We have expressed 2^n-1 as a product of two integers. Let us check that both are greater than 1:

- $2^a - 1 \geq 2^2 - 1 = 3$, since $a \geq 2$.

- The second factor is a sum of b positive terms, each at least 1, so it is at least $b \geq 2$.

So $2^n - 1$ is the product of two integers, each bigger than 1, and so $2^n - 1$ is not prime.

This proves the contrapositive, and therefore the original statement. □

What to take away. The contrapositive trades the original question for a slightly different one. Here the original asks us to say something about when $2^n - 1$ is prime, and primes don't hand over their structure. The contrapositive asks us to say something about when n is composite, which is easy, since we can just write $n = ab$. That shift is the whole trick.

A warning: the converse "if n is prime, then $2^n - 1$ is prime" is false. For instance $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23 \times 89$ is not. So the property "n prime" is necessary for $2^n - 1$ to be prime, but it is not enough.

Proof by contradiction

Problem 1.3. Show that $\log_2 3$ is irrational.

Technique. Proof by contradiction: assume the thing you want to disprove, and derive something impossible.

Discussion. Suppose, for a contradiction, that $\log_2 3$ is rational. Then it can be written as p/q for some positive integers p and q (both positive because $\log_2 3$ is a positive number, being greater than 1). By the definition of logarithm,

$$2^{p/q} = 3, \qquad \text{and so} \qquad 2^p = 3^q.$$

Now look at the two sides. The left side, 2^p, is a power of 2, so it is even (since $p \geq 1$). The right side, 3^q, is a power of 3, so it is odd. An integer cannot be even and odd at the same time. Contradiction.

Therefore $\log_2 3$ cannot be rational. □

What to take away. Proof by contradiction is the only tool that lets you assume something whose truth you do not yet know. You assume it, chase the consequences, and hope to reach nonsense. Here the nonsense was “an integer is both even and odd”. Once the nonsense appears, the assumption must have been wrong.

Problem 1.4. Show that there are infinitely many prime numbers.

Technique. Contradiction together with an explicit construction: assume the list is finite, then build an integer the list cannot explain.

Discussion. This is Euclid’s argument, from around 300 BC.

Suppose, for a contradiction, that there are only finitely many primes, and list them all: $p_1, p_2, \ldots, p_k$. Form the new integer

$$N = p_1 p_2 \cdots p_k + 1.$$

$N > 1$, so N has some prime factor; call it p. Is p on the list? If $p = p_i$ for some i, then p_i divides both N and $p_1 p_2 \cdots p_k$, so p_i must also divide their difference, $N - p_1 p_2 \cdots p_k = 1$. But no prime divides 1. So p is *not* on the list.

But we assumed the list contained every prime. Contradiction. Therefore the list of primes cannot be finite. □

What to take away. Notice the shape of the argument: assume the thing exists (a finite list), then *build* an object (the number N) that cannot fit. The construction $N = p_1 p_2 \cdots p_k + 1$ is the creative step; without it, we would just be staring at an assumption. Many impossibility proofs follow the same template.

Proof by cases

Problem 1.5. Show that no integer of the form $4k + 3$ (where k is an integer) can be written as a sum of two perfect squares.

Technique. Proof by cases on remainders: check each possible remainder on division by a well-chosen number.