Select Problems from Math Olympiads around the World

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1 BMO 1971

1.1 Round 1 Problem 7

Two real numbers h and k are given, such that $h > k > 0$. Find the probability that two points are chosen at random on a straight line of length h should be at a distance of less than k apart.

1.1.1 Solution

The required probability is $\frac{A}{h^2}$ where A is area between the lines $y = x + k$ and $y = x - k$ that lies inside a square of side h. The required probability is $\frac{k(2h-k)}{h^2}$.

1.1.2 Computational verification

```
import numpy as np

def simulate_point_distance_probability(n_simulations=1000000,
threshold=1/3):
    point1 = np.random.random(n_simulations)
    point2 = np.random.random(n_simulations)
    distances = np.abs(point2 - point1)
    favorable_outcomes = np.sum(distances < threshold)
    probability = favorable_outcomes / n_simulations
    return probability

probability = simulate_point_distance_probability()
print(f"Estimated probability: {probability:.4f}")
```

1.2 Round 1 Problem 9

Two uniform solid spheres of equal radii ars so placed that one is directly above the other. The bottom sphere is fixed and the top sphere, initially at rest, rolls off. If the coefficient of friction between the two spherical surfaces is μ, show that slipping occurs when $2\sin\theta = \mu(17\cos\theta - 10)$, where θ is the angle that the line of centres makes with the vertical.

1.2.1 Solution

From conservation of energy, we have

$$\begin{aligned} mg2r(1-\cos\theta) &= \frac{1}{2}(r\omega)^2 + \frac{1}{2}\cdot\frac{2}{5}mr^2\omega^2 \\ \Rightarrow 20g(1-\cos\theta) &= 7r\omega^2 \end{aligned} \tag{1.1}$$

Equating forces through the center of the top sphere, we have

$$N + m\frac{(r\omega)^2}{2r} = mg\cos\theta \tag{1.2}$$

Equating angular forces around the center of the top sphere, we have

$$rf_r = \frac{2}{5}mr^2\dot{\omega} \Rightarrow f_r = \frac{2}{5}mr\dot{\omega} \tag{1.3}$$

Just when the top sphere is about to slip, we have

$$f_r = \mu N \tag{1.4}$$

Equating forces parallel to the point of tangency, we have

$$mg\sin\theta - f_r = mr\dot{\omega} \tag{1.5}$$

From all the above, we get

$$2\sin\theta = \mu(17\cos\theta - 10). \tag{1.6}$$

2 BMO 1972

2.1 Round 1 Problem 9

A vertical uniform rod of length $2a$ is hinged at its lowest end to a frictionless joint secured to a horizontal table. It falls from rest in this unstable position on to the table. Find the time occupied in falling.

2.1.1 Solution

When the rod is at an angle θ with respect to the vertical line passing through the joint:

From conservation of energy we have,

$$\begin{aligned} mga(1-\cos(\theta)) &= \frac{1}{2}m\frac{4a^2}{3}\omega^2 \\ \Rightarrow \frac{3g}{2a}(1-\cos(\theta)) &= \omega^2 \\ \Rightarrow \sqrt{\frac{3g}{a}}\sin\left(\frac{\theta}{2}\right) &= \frac{d\theta}{dt} \\ \Rightarrow 2\log\left(\tan\left(\frac{\theta}{4}\right)\right) &= \sqrt{\frac{3g}{a}}t + c \end{aligned} \tag{2.1}$$

with the boundary condition $\theta(0) = 0$.

2.2 Round 1 Problem 10

A right circular cone the vertex of which is V and the semi vertical angle of which is α has height h and uniform density. All points of the cone the distances of which from V are less than a or greater than b, where $0 < a < b < h$, are removed, A solid of mass M is left. Gven that the gravitational attraction that a point mass m at P exerts on a unit mass at O is $(G\frac{m}{O}P^3)\overrightarrow{OP}$, prove that the magnitude of the gravitational attraction of this solid on a unit mass at V is

$$\frac{3}{2}\frac{GM(1+\cos\alpha)}{a^2+b^2+ab}. \tag{2.1}$$

2.2.1 Solution

3 BMO 1982

3.1 Round 1 Problem 6

Prove that the number of sequences $a_1, a_2, ..., a_n$ with each $a_i = 0$ or 1 and containing exactly m occurrences of '01', is $\binom{n+1}{2m+1}$.

3.1.1 Solution

A bit strings with m occurrences of 01 can be treated as a concatenation of $m+1$ bit strings where m of them end in 01. The first m bit strings are of the form 1^*0^+1 which means zero or more 1s and one or more 0s, and the last one is of the form 1^*0^*. Let x_{2i-1} be the number of $1s$ in the i^{th} bit string except the last 1 where $i = 1, ..., m$ and x_{2m+1} be the number of $1s$ in the $(m+1)^{st}$ bit string. Let x_{2i} be the number of $0s$ in the i^{th} where $i = 1, ..., m+1$. We have

$$x_1 + x_2 + ... + x_{2m+2} = n - m. \tag{3.1}$$

where $x_{2i} > 0$ for $i = 1, ..., m$, $x_{2j-1} \geq 0$ for $j = 1, ..., m+1$ and $x_{2m+2} \geq 0$ The number of solutions to the above equation is equal to the number of bit strings which satisfy the constraints of the problem. The number of solutions is therefore $\binom{n-m+(m+2)-1}{2m+2-1} = \binom{n+1}{2m+1}$.

3.1.2 Computational solution

Here is an elegant recursive code to calculate the number of bit strings.

```
def f(n, m, b):
    if n == 2:
        if m == 1 and b == 1:
            return 1
        elif m == 1 and b == 0:
            return 0
        elif m == 0 and b == 0:
            return 2
        elif m == 0 and b == 1:
            return 1
        return 0

    if m < 0:
        return 0

    if b == 0:
        return f(n-1, m, 0) + f(n-1, m, 1)
```

```
    else:
        return f(n-1, m, 1) + f(n-1, m-1, 0)

result = f(10, 4, 0) + f(10, 4, 1)
print(f"Number of bit strings of length 10 with exactly 3 occurrences
of '01': {result}")
```

4 BMO 1993

4.1 Round 2 Problem 3

Let $m = \frac{4^p-1}{3}$, where p is a prime number exceeding 3. Prove that 2^{m-1} has remainder 1 when divided by m.

4.1.1 Solution

If $m = \frac{4^p-1}{3}$, then

$$m = \frac{(2^p-1)(2^p+1)}{3} \tag{4.1}$$

We have,

$$\begin{aligned} 2^{m-1} - 1 &= 2^{\frac{4^p-4}{3}} - 1 \\ &= \left(2^{\left(\frac{4^{p-1}-1}{3}\right)}\right)^4 - 1 \\ &= \left(2^{\left(\frac{4^{p-1}-1}{3}\right)} - 1\right)\left(2^{\left(\frac{4^{p-1}-1}{3}\right)} + 1\right)\left(2^{2\left(\frac{4^{p-1}-1}{3}\right)} + 1\right) \end{aligned} \tag{4.2}$$

The first factor of $2^{m-1} - 1$ is,

$$2^{\left(\frac{4^{p-1}-1}{3}\right)} - 1 = \left(2^{\frac{2^{p-1}-1}{3}}\right)^{(2^{p-1}+1)} - 1 \tag{4.3}$$

is divisible by first factor of m because $2^{p-1} - 1$ is divisible by both p and 3.

The second factor of $2^{m-1} - 1$ is

$$2^{\left(\frac{4^{p-1}-1}{3}\right)} + 1 = \left(2^{\frac{2^{p-1}-1}{3}}\right)^{(2^{p-1}+1)} + 1 \tag{4.4}$$

is divisible by $2^p + 1$ because $2^{p-1} - 1$ is odd and divisible by both p and 3 and $2^{p-1} + 1$ is also an odd number.

As $2^{m-1} - 1$ is perfectly divisible by m, 2^{m-1} leaves a remainder 1 when divided by m.

We made use of the following facts:

(i) **Fermat's Little Theorem (FLT)** which states $a^{p-1} - 1$ is divisible by p when p is prime and p doesn't divide a.

(ii) $a^n - b^n$ is divisible by $a - b$ for all n.

(iii) $a^n + b^n$ is divisible by $a + b$ if n is odd.

5 BMO 1994

5.1 Round 2 Problem 1

Find the first integer $n > 1$ such that the average of $1^2, 2^2, 3^2, ..., n^2$ is itself a perfect square.

5.1.1 Solution

Average of the squares of the first n natural numbers is $\frac{(n+1)(2n+1)}{6}$.

For the average to be a perfect square, $(n+1)(2n+1) = 6k^2$ for some natural $k > 1$. Therefore n has to be odd.

If $n = 2m + 1$ for some natural m, we have

$$(m+1)(4m+3) = 3k^2 \tag{5.1}$$

Rearranging the above, we get

$$m(4m+7) = 3(k^2 - 1) \tag{5.2}$$

Therefore $m = 3t$ for some natural $t > 1$ and $n = 6t + 1$.

Rearranging Equation (5.2), we get $(3t+1)(4t+1) = k^2$.

$3t + 1$ and $4t + 1$ are relatively prime, so $3t + 1$ and $4t + 1$ should be perfect squares.

$$\begin{aligned} 3t + 1 &= x^2 \\ 4t + 1 &= y^2 \end{aligned} \tag{5.3}$$

where x and y are natural numbers.

From the above two equations, we get an equation of the form

$$4x^2 - 3y^2 = 1 \tag{5.4}$$

The above equation can also be written as

$$\left(2x - \sqrt{3}y\right)\left(2x + \sqrt{3}y\right) = \left(2 - \sqrt{3}\right)^p\left(2 + \sqrt{3}\right)^p \tag{5.5}$$

for some natural p which gives rise to the equations

$$\begin{aligned} 2x - \sqrt{3}y &= \left(2 - \sqrt{3}\right)^p \\ 2x + \sqrt{3}y &= \left(2 + \sqrt{3}\right)^p \end{aligned} \tag{5.6}$$

Solving the system of equations, we get

$$
\begin{aligned}
x &= \frac{\left(2-\sqrt{3}\right)^p + \left(2+\sqrt{3}\right)^p}{4} \\
y &= \frac{\left(2+\sqrt{3}\right)^p - \left(2-\sqrt{3}\right)^p}{2\sqrt{3}}
\end{aligned}
\tag{5.7}
$$

We can quickly check that $p = 3$ is the smallest p for which both $x = 13$ and $y = 15$ are integers. When $p = 3$, we get $t = 56$ and $n = 337$.

6 BMO 1995

6.1 Round 2 Problem 4

(a) Determine how many ways $2n$ people can be paired off to form n teams of 2.

(b) Prove that $(mn!)^2$ is divisible by $(m!)^{n+1}(n!)^{m+1}$ for all positive integers m, n.

6.1.1 Solution

For (a), the first 2 people can be selected in $\binom{2n}{2}$ ways, the next 2 people in $\binom{2n-2}{2}$ ways and so on. Therefore, the total number of ways in which n teams of 2 can be chosen from $2n$ people is

$$\prod_{i=1}^{n}\binom{2n-2(i-1)}{2} = \frac{2n!}{2!^n} \tag{6.1}$$

The order in which the n teams are formed is irrelevant, so the total number of ways of forming n teams of 2 from $2n$ people is

$$\frac{2n!}{2!^n \cdot n!} \tag{6.2}$$

To prove (b), using the same logic as above, we can see that the number of ways in which n teams of m people can be formed from mn people is

$$\frac{mn!}{m!^n \cdot n!} \tag{6.3}$$

Similiarly, the number of ways in which m teams of n people can be formed from mn people is

$$\frac{mn!}{n!^m \cdot m!} \tag{6.4}$$

Multiplying the above two integers we get

$$\frac{mn!^2}{n!^{m+1} \cdot m!^{n+1}} \tag{6.5}$$

which proves that $mn!^2$ is divisible by $m!^{n+1} \cdot n!^{m+1}$ for all positive integers m, n.

7 BMO 1996

7.1 Round 1 Problem 2

A function f is defined over the set of all positive integers and satisfies $f(1) = 1996$ and $f(1) + f(2) + \ldots + f(n) = n^{\{2\}} f(n)$ for all $n > 1$. Calculate the exact value of $f(1996)$.

7.1.1 Solution

We have

$$\begin{aligned} f(n+1) &= (n+1)^2 f(n+1) - n^2 f(n) \\ &= n^2((n+1)^2 - 1) f(n) \\ &= \left(\frac{n}{n+2}\right) f(n) \end{aligned} \tag{7.1}$$

Therefore,

$$\begin{aligned} \frac{f(2)}{f(1)} \cdot \frac{f(3)}{f(2)} \ldots \frac{f(1996)}{f(1995)} &= \frac{1}{3} \cdot \frac{2}{4} \ldots \frac{1995}{1997} \\ \Rightarrow \frac{f(1996)}{f(1)} &= \frac{1}{1996} \cdot \frac{2}{1997} \\ \Rightarrow f(1996) &= \frac{2}{1997} \end{aligned} \tag{7.2}$$

7.1.2 Verification using code

We have $f(n) = \frac{\sum_{i=1}^{n-1} f(i)}{n^2-1}$.

```
from fractions import Fraction
n = 1996
sumfn = Fraction(1996,1)
for i in range(2, n+1):
  fn_next = sumfn/(i**2-1)
  sumfn += fn_next
print("f(%d) = %s" % (n,fn_next))
```

7.2 Round 1 Problem 5

Let a, b and c be positive real numbers.

(i) Prove that $4(a^3 + b^3) \geq (a+b)^3$

(ii) Prove that $9(a^3 + b^3 + c^3) \geq (a+b+c)^3$

7.2.1 Solution

For (i), we have

$$a^3 + b^3 \geq a^3 + b^3 \tag{7.1}$$

$$3(a^3 + b^3) \geq 3(ab^2 + ba^2) \tag{7.2}$$

We get Equation (7.2) from Muirhead's inequality because $(3, 0)$ majorizes $(2, 1)$. Adding the above inequalities, we get (i).

For (ii), we have

$$a^3 + b^3 + c^3 \geq a^3 + b^3 + c^3 \tag{7.3}$$

$$6(a^3 + b^3 + c^3) \geq 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) \tag{7.4}$$

$$2(a^3 + b^3 + c^3) \geq 6(abc) \tag{7.5}$$

We get Equation (7.4) from Muirhead's inequality because $(3, 0)$ majorizes $(2, 1)$.

We have Equation (7.5), because $AM \geq GM$.

Adding all the above inequalities, we get (ii).

8 BMO 1997

8.1 Round 1 Problem 2

For positive integers n, the sequence $a_1, a_2, a_3, ..., a_n, ...$ is defined by

$$a_1 = 1; a_n = \frac{n+1}{n-1}(a_1 + a_2 + a_3 + ... + a_{n-1}), n > 1 \tag{8.1}$$

Determine the value of a_{1997}.

8.1.1 Solution 1

We have

$$\begin{aligned} a_{n+1} &= a_n \frac{2(n+2)}{n+1} \\ \Rightarrow \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} ... \frac{a_n}{a_{n-1}} &= a_n = 2^{n-2}(n+1) \end{aligned} \tag{8.2}$$

8.1.2 Brute force computational solution

```
n = 5
sumfn = 1
for i in range(2, n+1):
  fn_next = (sumfn*(i+1))/(i-1)
  sumfn += fn_next
print("f(%d) = %s" % (n,fn_next))
```

8.2 Round 2 Problem 3

Find the number of polynomials of degree 5 with distinct coefficients from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are divisible by $x^2 - x + 1$.

8.2.1 Solution

A fifth degree polynomial is of the form

$$P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f. \tag{8.1}$$

It is easy to see that $-\omega$ and $-\omega^2$ are the roots of $x^2 - x + 1$.

Therefore for $x^2 - x + 1$ to divide $P(x)$, we need

$$\begin{aligned} P(-\omega) &= -a\omega^2 + b\omega - c + d\omega^2 - e\omega + f = 0 \\ P(-\omega^2) &= -a\omega + b\omega^2 - c + d\omega - e\omega^2 + f = 0 \end{aligned} \tag{8.2}$$

From the above two equations, we get

$$\boldsymbol{a + b = d + e} \tag{8.3}$$

$$\boldsymbol{b + c = e + f} \tag{8.4}$$

9 BMO 1999

9.1 Round 2 Problem 3

Non-negative real numbers p, q and r satisfy $p + q + r = 1$. Prove that

$$7(pq + qr + rp) \leq 2 + 9pqr. \tag{9.1}$$

9.1.1 Solution

$$\begin{aligned}
2 + 9pqr &= 2((p + q + r)^3) + 9pqr \\
&= 2(p^3 + q^3 + r^3) \\
&+6(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) + 21pqr
\end{aligned} \tag{9.2}$$

$$\begin{aligned}
7(pq + qr + rp) &= 7(pq + qr + rp)(p + q + r) \\
&= 7(p^2q + pq^2 + pqr + pqr \\
&+q^2r + qr^2 + rp^2 + pqr + r^2p) \\
&= p^2q + p^2r + q^2p + q^2r + r^2p + r^2q + \\
&6(p^2q + p^2r + q^2p + q^2r + r^2p + r^2q) + 21pqr
\end{aligned} \tag{9.3}$$

From Muirhead's inequality we have

$$2(p^3 + q^3 + r^3) \geq p^2q + p^2r + q^2p + q^2r + r^2p + r^2q \tag{9.4}$$

because $(3, 0, 0)$ majorizes $(2, 1, 0)$.

Therefore $7(pq + qr + rp) \leq 2 + 9pqr$.

10 BMO 2000

10.1 Round 1 Problem 2

Show that, for every positive integer n,

$$121^n - 25^n + 1900^n - (-4)^n \tag{10.1}$$

is divisible by 2000.

10.1.1 Solution

As $a^n - b^n$ is divisible by $a - b$ for all n,

$121^n - (-4)^n$ is divisible by $121 - (-4) = 125$.

$1900^n - 25^n$ is divisible by $1900 - 25 = 1875$ which is divisible by 125.

From the above we see that $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 125.

We also have

$$\begin{aligned} 121 \equiv 1 \bmod 8 &\Rightarrow 121^n \equiv 1 \bmod 8 \\ 25 \equiv 1 \bmod 8 &\Rightarrow 25^n \equiv 1 \bmod 8 \\ 1900 \equiv 4 \bmod 8 &\Rightarrow 1900^n \equiv 4^n \bmod 8 \\ -4 \equiv 4 \bmod 8 &\Rightarrow (-4)^n \equiv 4^n \bmod 8 \end{aligned} \tag{10.2}$$

From the above we see that $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 8. As $121^n - 25^n + 1900^n - (-4)^n$ is divisible by 125 and 8 which are relatively prime to each other, it is divisible by $125 \cdot 8 = 2000$.

10.2 Round 2 Problem 2

Given that x, y, z are positive real numbers satisfying $xyz = 32$, find the minimum value of

$$x^2 + 4xy + 4y^2 + 2z^2 \tag{10.1}$$

10.2.1 Solution

Using $AM \geq GM$ as all the quantities involved are positive, we have

$$\begin{aligned} \frac{x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2}{6} &\geq \left(x^2 \cdot 2xy \cdot 2xy \cdot 4y^2 \cdot z^2 \cdot z^2\right)^{\frac{1}{6}} = 16 \\ \Rightarrow x^2 + 4xy + 4y^2 + 2z^2 &\geq 96 \end{aligned} \tag{10.2}$$

Therefore, the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$ is 96.

11 BMO 2001

11.1 Round 1 Problem 5

A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

11.1.1 Solution

In a triangle

$$\frac{a}{sin(A)} = \frac{b}{sin(B)} = \frac{c}{sin(C)} = 2R \tag{11.1}$$

In a right-angled triangle with $\angle C = 90°$

$$\begin{aligned} a^2 + b^2 + c^2 &= 4R^2 sin^2(A) + 4R^2 sin^2(B) + 4R^2 \\ &= 4R^2 sin^2(A) + 4R^2 cos^2(A) + 4R^2 \\ &= 8R^2 \end{aligned} \tag{11.2}$$

12 BMO 2002

12.1 Round 1 Problem 1

Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that $x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}$.

12.1.1 Solution

Using **Muirhead's** inequality, we have

$$2(x^2yz + xy^2z + xyz^2) \leq 2(x^4 + y^4 + z^4) \tag{12.1}$$

$$2(x^2yz + xy^2z + xyz^2) \leq 2x^2y^2 + 2x^2z^2 + 2y^2z^2 \tag{12.2}$$

We have the first inequality because $(4, 0, 0)$ majorizes $(2, 1, 1)$.

We have the second because $(2, 2, 0)$ majorizes $(2, 1, 1)$.

Adding the above inequalities, we get

$$3(x^2yz + xy^2z + xyz^2) \leq (x^2 + y^2 + z^2)^2 = 1 \tag{12.3}$$

13 BMO 2005

13.1 Round 2 Problem 3

Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{13.1}$$

13.1.1 Solution

Expanding the left side, we get

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+2\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right) \tag{13.2}$$

Expanding the right side, we get

$$3+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+\frac{b}{c}+\frac{c}{a}+\frac{c}{b} \tag{13.3}$$

Therefore, to prove the original inequality it is sufficient to prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \tag{13.4}$$

because from $AM \geq GM$ we see that

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b} \geq 3 \tag{13.5}$$

Inequality Equation (13.4) is equivalent to the following:

$$\begin{gathered} xyz = 1 \\ x^2+y^2+z^2 \geq x+y+z \end{gathered} \tag{13.6}$$

where $x = \frac{a}{b}$, $y = \frac{b}{c}$ and $z = \frac{c}{a}$.

Combining the above inequalities, we get

$$x^2+y^2+z^2 \geq x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}(x+y+z) \tag{13.7}$$

Since $(2,0,0) \succ \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right)$, applying Muirhead's inequality solves the problem.

14 BMO 2006

14.1 Round 1 Problem 1

Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

14.1.1 Solution

Factorizing,we have

$$\begin{aligned} 3^{32} - 2^{32} &= (3^{16} + 2^{16})(3^{16} - 2^{16}) \\ &= (3^{16} + 2^{16})(3^{8} + 2^{8})(3^{4} + 2^{4})(3^{2} + 2^{2})(3^{2} - 2^{2}) \end{aligned} \quad (14.1)$$

From Fermat's Little Theorem, we have $3^{17-1} - 1 - (2^{17-1} - 1)$ is divisible by 17.

From Equation (14.1), we see that $(3^{\{32\}} - 2^{\{32\}})$ is divisible by 5, 13 and 97.

15 BMO 2010

15.1 Round 1 Problem 1

One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

15.1.1 Solution

Let the integer removed be x. The average of the remaining numbers is

$$\begin{aligned} \frac{n(n+1)}{2(n-1)} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \Rightarrow \frac{(n-1)^2 + 3(n-1) + 2}{2(n-1)} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \Rightarrow \frac{n-1}{2} + \frac{3}{2} + \frac{1}{n-1} - \frac{x}{n-1} &= 40\frac{3}{4} \\ \Rightarrow \frac{n-1}{2} = 40 \wedge \frac{x-1}{n-1} &= \frac{3}{4} \end{aligned} \tag{15.1}$$

From the above we get $n = 81$ and $x = 61$.

16 BMO 2012

16.1 Round 1 Problem 4

Find all positive integers n such that $12n - 119$ and $75n - 539$ are both perfect squares.

16.1.1 Solution

Let,

$$\begin{aligned} 12n - 119 &= x^2 \\ 75n - 539 &= y^2 \end{aligned} \tag{16.1}$$

From the two equations above, we have

$$\begin{aligned} 4y^2 - 25x^2 &= (2y - 5x)(2y + 5x) \\ &= 3 \cdot 273 = 9 \cdot 91 = 39 \cdot 21 = 13 \cdot 63 \end{aligned} \tag{16.2}$$

Solving the sets of simultaneous equations resulting from the above equation and restricting ourselves to integer solutions, we get the following solutions $(x, y) = (27, 69)$ and $(x, y) = (5, 19)$. When $x = 5$ we get $n = 12$. For $x = 27$, n is not an integer.

17 BMO 2016

17.1 Round 1 Problem 3

Determine all pairs (m, n) of positive integers which satisfy the equation

$$n^2 - 6n = m^2 + m - 10 \tag{17.1}$$

17.1.1 Solution

Completing squares on both sides we get

$$\begin{aligned} (n-3)^2 - 9 &= \left(m + \frac{1}{2}\right)^2 - \frac{1}{4} - 10 \\ \Rightarrow (2m+1)^2 - (2n-6)^2 &= 5 \end{aligned} \tag{17.2}$$

From the above, we get the following sets of simultaneous equations,

$$\begin{aligned} 2m - 2n = -6 &\vee 2m - 2n = -2 \\ 2m + 2n = 10 &\vee \ 2m + 2n = 6 \end{aligned} \tag{17.3}$$

Solving the above, we get $(m, n) = (1, 4)$ or $(m, n) = (1, 2)$.

18 CMO 1969

18.1 Problem 7

Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

18.1.1 Solution

Every square integer is of the form $8k$, $8k + 1$ or $8k + 4$.

It is sufficient to consider the following 6 cases

Form of a^2	Form of b^2	Form of $a^2 + b^2 - 8c$
$8s$	$8t$	$8(s + t - c)$
$8s$	$8t + 1$	$8(s + t - c) + 1$
$8s + 1$	$8t + 1$	$8(s + t - c) + 2$
$8s$	$8t + 4$	$8(s + t - c) + 4$
$8s + 1$	$8t + 4$	$8(s + t - c) + 5$
$8s + 4$	$8t + 4$	$8(s + t - c + 1)$

In each of the cases, it is easy to see that $a^2 + b^2 - 8c$ can never be 6.

19 CMO 1971

19.1 Problem 2

Let x and y be positive real numbers such that $x + y = 1$. Show that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9 \tag{19.1}$$

19.1.1 Solution

By $AM \geq GM$, we have

$$\frac{x + x + y}{3} \cdot \frac{y + x + y}{3} \geq (x^2 y)^{\frac{1}{3}} (y^2 x)^{\frac{1}{3}} = xy \tag{19.2}$$

Replacing $x + y$ by 1, multiplying both sides by 9 and dividing by xy, we get the required inequality.

20 CMO 1978

20.1 Problem 3

Determine the largest real number z such that

$$\begin{aligned} x + y + z &= 5 \\ xy + yz + xz &= 3 \end{aligned} \tag{20.1}$$

and x, y are also real.

20.1.1 Solution

As x and y are real, $(x + y)^2 \geq 4xy$ which implies

$$\begin{aligned} (5 - z)^2 &\geq 4(3 - z(5 - z)) \\ \Rightarrow 3z^2 - 10z - 13 &\leq 0 \\ \Rightarrow -1 \leq z &\leq \frac{13}{3} \end{aligned} \tag{20.2}$$

21 CMO 1991

21.1 Problem 2

Let n be a fixed positive integer. Find the sum of all positive integers with the following property: In base 2, it has exactly $2n$ digits consisting of n 1's and n 0's.(The first digit cannot be 0).

21.1.1 Solution

The number of numbers with $2n$ binary digits where the binary form of the number contains 1 in the first position and $(n-1)$ $1's$ in the rest of the $2n-1$ positions is $\binom{2n-1}{n-1}$. Each number can be represented as follows:

$$2^{2n-1} + b_{2n-1}2^{2n-2} + \dots + b_1 \tag{21.1}$$

Out of $\binom{2n-1}{n-1}$ numbers, each $b_i = 1$ for $i \in \{1, 2, \dots, 2n-1\}$ in $\binom{2n-1}{n-1}\frac{n-1}{2n-1} = \binom{2n-2}{n}$ cases.

Therefore some of all the numbers which are of the form described in the first equation is given by

$$\begin{aligned} &\binom{2n-1}{n-1}2^{2n-1} + \binom{2n-2}{n}(1 + 2 + \dots + 2^{2n-2}) \\ &= \binom{2n-1}{n}2^{2n-1} + \binom{2n-2}{n}(2^{2n-1} - 1) \end{aligned} \tag{21.2}$$

22 CMO 1992

22.1 Problem 4

Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3 \tag{22.1}$$

22.1.1 Solution

Let $y = x + 1$, then the above equation is equivalent to

$$\begin{aligned} & (y-1)^2 + \frac{(y-1)^2}{y^2} = 3 \\ \Rightarrow\ & y^2 - 2y + 1 + 1 - \frac{2}{y} + \frac{1}{y^2} = 3 \\ \Rightarrow\ & \left(y + \frac{1}{y}\right)^2 - 2\left(y + \frac{1}{y}\right) = 3 \\ \Rightarrow\ & \left(y + \frac{1}{y} + 1\right)\left(y + \frac{1}{y} - 3\right) = 0 \\ \Rightarrow\ & y^2 + y + 1 = 0 \vee y^2 - 3y + 1 = 0 \end{aligned} \tag{22.2}$$

Therefore $x = \omega - 1$ or $x = \omega^2 - 1$ or $x = \frac{1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}$ where ω is the cube root of unity.