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Subsets of $\{1,2,\dots,n\}$ Containing Exactly One Pair of Consecutive Integers

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October 2024

Problem

Find the number of subsets of $\{1,2,3,\dots,n\}$ that contain exactly one pair of consecutive integers. Examples of such subsets are $\{1,2,5\}$ and $\{1,3,6,7,10\}$.

Solution by recurrences and generating functions

There are numerous ways to solve this problem; here is one involving generating functions.

Subsets with no consecutive integers

First, find the number of subsets of $\{1,2,\dots,n\}$ for $n \geq 2$ with no consecutive integers. This is equivalent to counting bit strings of length n with no consecutive ones. Let $g(n)$ denote this count.

A bit string with the required property starts either with a one (in which case the next bit must be a zero, and the remaining $n-2$ bits form an admissible string) or with a zero (in which case the remaining $n-1$ bits form one). Thus

$$g(n) \;=\; g(n-1)+g(n-2), \qquad g(2)=3, \quad g(3)=5.$$

The recurrence is Fibonacci, so $g(n)=F_{n+2}$.

Subsets with one pair of consecutive integers

Equivalently, count bit strings of length n with exactly one pair of consecutive ones. Let $f(n)$ denote this count.

A string with the property starting with two consecutive ones has its remaining $n-2$ bits form a string with no consecutive ones, giving $g(n-2)=F_n$ choices. By symmetry, the same count applies to strings ending in two consecutive ones, for a total of $2F_n$.

A string with exactly one pair of consecutive ones in the middle is formed by concatenating two strings with no consecutive ones, the left ending in a one and the right beginning with a one. With the left of length $i \geq 2$ and the right of length

$n - i - 2 \geq 2$, this contributes $g(i-2)\, g(n-i-4) = F_i F_{n-i-2}$ for each interior split. Summing,

$$f(n) \;=\; \sum_{i=2}^{n-2} F_i\, F_{n-i} \;+\; 2\, F_{n-1} \;=\; \sum_{i=1}^{n-1} F_i\, F_{n-i}.$$

Closed form via generating functions

Start from the Fibonacci generating function

$$F(x) \;=\; \sum_{n \geq 1} F_n\, x^n \;=\; \frac{x}{1 - x - x^2}.$$

The convolution above is the coefficient of x^n in $F(x)^2$:

$$f(n) \;=\; [x^n]\, \frac{x^2}{(1 - x - x^2)^2} \;=\; [x^{n-2}]\, \frac{1}{(1 - x - x^2)^2}.$$

Decompose $1/(1 - x - x^2)^2$ by partial fractions over the roots $r_\pm = (1 \pm \sqrt{5})/2$ of $1 - x - x^2$ (so $r_+ r_- = -1$ and $r_+ + r_- = 1$):

$$\frac{1}{(1 - x - x^2)^2} \;=\; \frac{1}{(r_+ - r_-)^2}\left(\frac{r_+^2}{(1 - x r_+)^2} + \frac{r_-^2}{(1 - x r_-)^2} - \frac{2\, r_+ r_-}{1 - x - x^2}\right).$$

Extract the coefficient of x^{n-2} using $[x^k](1 - rx)^{-2} = (k+1) r^k$. Collecting,

$$f(n) \;=\; \frac{n-1}{5} L_n + \frac{2}{5} F_{n-1},$$

where L_n is the n-th Lucas number. At $n = 10$:

$$f(10) \;=\; \tfrac{9 \cdot 123}{5} + \tfrac{2 \cdot 34}{5} \;=\; \mathbf{235}.$$

Computational verification

```
def has_exactly_one_consecutive_pair(subset):
    return sum(subset[i] == '1' and subset[i + 1] == '1'
               for i in range(len(subset) - 1)) == 1

def count_subsets_with_one_consecutive_pair(n):
    count_by_size = {i: 0 for i in range(1, n + 1)}
    total_count = 0
    for i in range(1, 2 ** n):
        subset = format(i, f'0{n}b')
        if has_exactly_one_consecutive_pair(subset):
            size = subset.count('1')
            count_by_size[size] += 1
            total_count += 1
    return total_count, count_by_size

n = 10
total, breakdown = count_subsets_with_one_consecutive_pair(n)
print(f"Total number of subsets with exactly one consecutive pair: {total}")
```

The brute-force enumeration confirms $f(10) = 235$.