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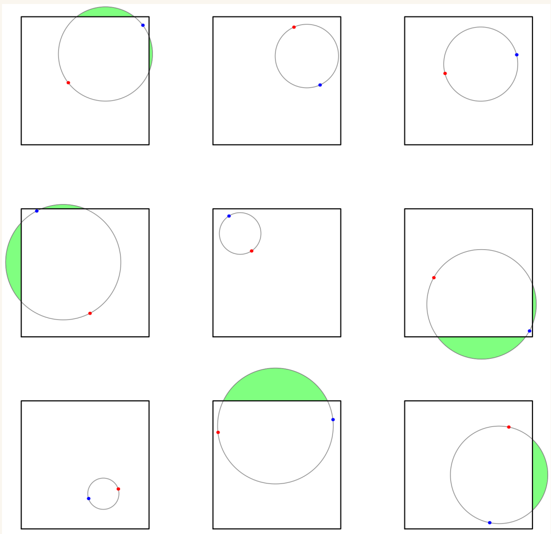
Some Off Square

October 2024

A geometric-probability puzzle.

Problem

A circle is generated at random by sampling two points uniformly and independently from the interior of a square and using these points as the endpoints of a diameter. What is the probability that part of the circle lies outside the square?



Solution

Let P_1 and P_2 be the two sampled points and M their midpoint. The random circle has centre M and radius $|MP_1| = |MP_2|$, and it crosses the boundary of the square exactly when the distance from M to that boundary is strictly less than the radius.

Take the square to be $[-1,1]^2$ and the two points to be $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ with each coordinate drawn independently from $\mathcal{U}[-1,1]$. The midpoint is $M = \frac{1}{2}(x_1 + x_2,\, y_1 + y_2)$ and its distance to the nearest side of the square is

$$\min\left(1 - \left|\frac{x_1+x_2}{2}\right|,\ 1 - \left|\frac{y_1+y_2}{2}\right|\right).$$

The radius is half the chord length:

$$\frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The event of interest is therefore

$$\min\Big(1 - |\tfrac{x_1+x_2}{2}|,\ 1 - |\tfrac{y_1+y_2}{2}|\Big) \le \frac{1}{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Computational solution

A Monte Carlo simulation over ten million trials gives the probability as

$$p \approx \mathbf{0.476}.$$

Python code

```
from random import uniform
from math import sqrt

runs = 10_000_000
cnt = 0
for _ in range(runs):
    x_1, x_2, y_1, y_2 = (
        uniform(-1, 1), uniform(-1, 1),
        uniform(-1, 1), uniform(-1, 1),
    )
    if min(2 - abs(x_1 + x_2),
           2 - abs(y_1 + y_2)) \
       <= sqrt((x_1 - x_2)**2 + (y_1 - y_2)**2):
        cnt += 1
print(cnt / runs)
```