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A Hundred Random Chords

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Problem

Choose a point uniformly at random on a circle, choose a second the same way, and join them with a chord. Do this 100 times, giving 100 random chords. What is the expected number of intersection points?

Solution

Everything rests on one question: what is the chance that *two* random chords cross? A chord pair involves four independent uniform points on the circle, all distinct with probability 1. Condition on *where* the four points sit and forget which is which: by exchangeability, every assignment of the four positions to the four labels is equally likely, so the two points forming the first chord are equally likely to be any of the three ways of splitting the four positions into two pairs. Walking round the circle through positions $1, 2, 3, 4$, the three splittings pair the positions as 12|34, 14|23, and 13|24, and only the last, the interleaved one, makes the chords cross. Hence

$$\Pr(\text{two random chords cross}) = \frac{1}{3}.$$

A second route gives the same number and will earn its keep shortly. The first chord's endpoints cut the circumference into two arcs, and the fraction t of the circle on one side is uniform on $(0, 1)$. The second chord crosses the first exactly when its two endpoints land on opposite sides, which given t has probability $2t(1 - t)$, and

$$\int_0^1 2t(1-t)\,dt = \frac{1}{3}.$$

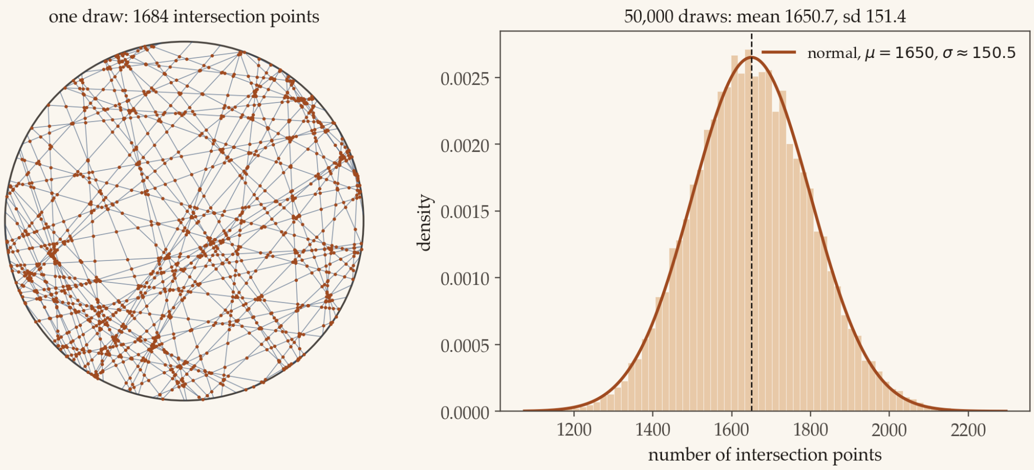
The rest is linearity of expectation. With probability 1 no two chords share an endpoint and no three pass through a common point, so the intersection points are exactly the crossing pairs of chords, and the pairs are $\binom{100}{2} = 4950$ in number.

[1] Peter Winkler, *Monthly Mindbenders*, National Museum of Mathematics, June 2026, momath.org/mindbenders; shared by Zach Wissner-Gross in *The Fiddler*, 17 July 2026.

Each crosses with probability $\frac{1}{3}$, and expectations add whether or not the events are independent (they are not), so the expected number of intersection points is

$$\binom{100}{2} \cdot \frac{1}{3} = \boxed{\frac{4950}{3} = 1650.}$$

In general, n random chords produce $n(n-1)/6$ crossings on average.



Left: one draw of 100 *random chords; this one happens to have* 1684 *crossings. Right: the crossing count over* 50,000 *draws, hugging the normal curve with mean* 1650 *and standard deviation* 150.5.

How much does it fluctuate?

The mean needed no thought about dependence, but the spread does. Write $X = \sum_p I_p$, the sum over the 4950 chord pairs of the indicator that the pair crosses. Each indicator has variance $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$. Two pairs with no chord in common involve eight independent points, so their indicators are independent and contribute nothing. The interesting case is two pairs sharing a chord, say $\{i, j\}$ and $\{i, k\}$: given the common chord's arc fraction t, the other two chords cross it independently, each with probability $2t(1-t)$, so

$$\Pr(\text{both cross}) = \int_0^1 (2t(1-t))^2 \, dt = \frac{2}{15}, \qquad \text{Cov} = \frac{2}{15} - \frac{1}{9} = \frac{1}{45}.$$

The covariance is positive: a chord that cuts near-diametrically across the circle is easy to cross, and one that hugs the rim is hard, so crossings sharing a chord rise and fall together. There are $n(n-1)(n-2)$ ordered pairs of pairs with a common chord, and altogether

$$\text{Var}(X) = \binom{n}{2}\frac{2}{9} + n(n-1)(n-2)\,\frac{1}{45} \overset{n=100}{=} 1100 + 21560 = 22660,$$

a standard deviation of $\sqrt{22660} \approx 150.5$. The shared-chord terms dominate the variance twenty to one, yet the count still behaves like a textbook normal, as the histogram shows.

A remark on regions

The crossing count is doing more work than it appears to. Draw the chords one at a time: a new chord that meets c of the earlier ones passes through $c + 1$ of the existing regions and splits each, adding $c + 1$ regions. Starting from one region and summing, the disc ends up cut into exactly $1 + n + X$ pieces. So the expected number of regions the 100 chords carve the disc into is

$$1 + 100 + 1650 = 1751,$$

an exact consequence of the count just computed, not a separate calculation.

Python code

The check replays the construction with no shortcuts: draw the endpoint angles, test every pair of chords for interleaving, and count. The pair test also confirms the two probabilities the derivation leans on, $\frac{1}{3}$ for one crossing and $\frac{2}{15}$ for two crossings of a shared chord.

```python
import numpy as np
rng = np.random.default_rng(7)
n, T, C = 100, 50_000, 100              # chords, trials, chunk
iu = np.triu_indices(n, 1)
counts = []
for _ in range(T // C):
    th = rng.random((C, n, 2))          # endpoint angles (turns)
    a, b = th[..., 0], th[..., 1]
    L = (b - a) % 1.0                   # arc from a round to b
    p = (a[:, None, :] - a[:, :, None]) % 1.0 # j's endpoints seen from a_i
    q = (b[:, None, :] - a[:, :, None]) % 1.0
    cross = (p < L[:, :, None]) ^ (q < L[:, :, None])
    counts.append(cross[:, iu[0], iu[1]].sum(axis=1))
X = np.concatenate(counts)
print(f"mean {X.mean():.2f} (theory {100*99/6:.0f})")
print(f"var {X.var():.0f} (theory {100*99/9 + 100*99*98/45:.0f})")
print(f"sd {X.std():.1f}  (theory {np.sqrt(100*99/9 + 100*99*98/45):.1f})")
# pairwise checks: P(cross) = 1/3, P(two chords both cross a third) = 2/15
th = rng.random((2_000_000, 3, 2)); a, b = th[..., 0], th[..., 1]
L0 = (b[:, 0] - a[:, 0]) % 1.0
def crosses(k):
    pk = (a[:, k] - a[:, 0]) % 1.0; qk = (b[:, k] - a[:, 0]) % 1.0
    return (pk < L0) ^ (qk < L0)
c1, c2 = crosses(1), crosses(2)
print(f"P(cross) {c1.mean():.5f} (1/3 = {1/3:.5f}); "
      f"P(both cross same chord) {np.mean(c1 & c2):.5f} (2/15 = {2/15:.5f})")
# mean 1648.71 (theory 1650)
# var 22698   (theory 22660)
# sd  150.7   (theory 150.5)
# P(cross) 0.33276 (1/3 = 0.33333); P(both cross same chord) 0.13302 (2/15 = 0.13333)
```