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Maximising the Length of a Projectile Trajectory

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A puzzle from *Fiddler on the Proof*.

Problem

At what angle should you throw a projectile, subject to uniform gravity and ignoring air resistance, in order to maximise the length of its trajectory?

Solution

Without loss of generality, take the launch speed to be 1 m/s. Let θ be the angle of launch. Then

$$x = \cos\theta\, t, \qquad y = \sin\theta\, t - \tfrac{1}{2}gt^2.$$

The length of the trajectory up to time t is

$$L(t) = \int_0^t \sqrt{\dot{x}^2 + \dot{y}^2}\, dt.$$

The projectile returns to the ground at $t = 2\sin\theta/g$. Therefore

$$\begin{aligned} L(\theta) &= \int_0^{2\sin\theta/g} \sqrt{\cos^2\theta + (\sin\theta - gt)^2}\, dt \\ &= \int_0^{2\sin\theta/g} \sqrt{1 - 2\sin\theta\, gt + g^2t^2}\, dt \\ &= \frac{1}{g}\int_0^{2\sin\theta} \sqrt{1 - 2\sin\theta\, t + t^2}\, dt. \end{aligned}$$

Evaluating using the standard integral

$$\int \sqrt{ax^2 + bx + c}\, dx = \frac{(2ax+b)\sqrt{x(ax+b)+c}}{4a} - \frac{(b^2 - 4ac)\log\!\left(2\sqrt{a}\sqrt{x(ax+b)+c} + 2ax + b\right)}{8\,a^{3/2}},$$

gives, after simplification,

$$L(\theta) = \frac{1}{2g}\left[2\sin\theta + \cos^2\theta \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right].$$

Differentiating with respect to θ and setting the derivative to zero:

$$2\csc\theta \;=\; \log\left(\frac{1+\sin\theta}{1-\sin\theta}\right).$$

Solving numerically,

$$\theta \;\approx\; 0.9855\ \text{rad} \;=\; \mathbf{56.465}^\circ.$$