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Inversion: A Beautiful Idea from Geometry

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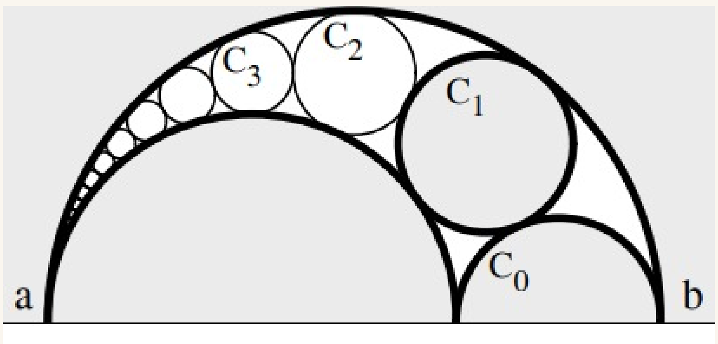
Inversion in a circle is one of those rare techniques that transforms a hard problem into an easy one by changing the ambient setting. The two problems below illustrate the move. The first is a Pappus-chain identity; the second is a coordinate computation for three mutually tangent circles. Both collapse once the right inversion centre is chosen.

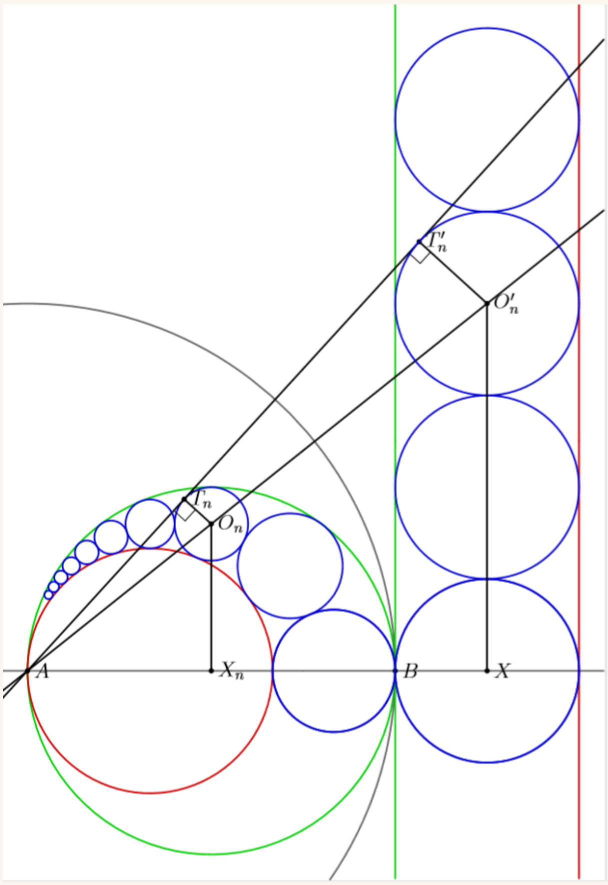
Problem 1: a Pappus chain

There are three semicircles and infinitely many circles inscribed in the figure. The radius of the smallest semicircle C_0 is $R_0 = \frac{1}{6}ab$. Prove that Y_n, the height of the centre of the n-th circle, equals $2nR_n$.



Solution

Use the circle whose centre is A and radius is AB as the circle of inversion.



The red and green circles, both passing through the centre of inversion A, get inverted to the red and green parallel lines. This follows from the standard properties of inversion:

1. Circles passing through the centre of inversion O invert to lines.
2. A line through O inverts to itself.
3. Two circles intersecting (or tangent) in k places invert to circles or lines intersecting in the same number of places.
4. A circle tangent to the inverting circle k inverts to a circle tangent to k at the same place.

Let O_n be the centre of the n-th circle C_n and O'_n its inverse. From similar triangles AO_nT_n and $AO'_nT'_n$,

$$\frac{O_nT_n}{AO_n} = \frac{O'_nT'_n}{AO'_n}.$$

Let X_n be the projection of O_n on AB and X the projection of O'_n on AB. Similar triangles AO_nX_n and AO'_nX give

$$\frac{O_nX_n}{AO_n} = \frac{O'_nX}{AO'_n}.$$

Combining the two,

$$\frac{O_nX_n}{O'_nX} = \frac{Y_n}{O'_nX} = \frac{O_nT_n}{O'_nT'_n} = \frac{R_n}{O'_nT'_n},$$

so

$$Y_n = R_n \cdot \frac{O'_n X}{O'_n T'_n} = R_n \cdot \frac{2nR'_0}{R'_0} = 2nR_n,$$

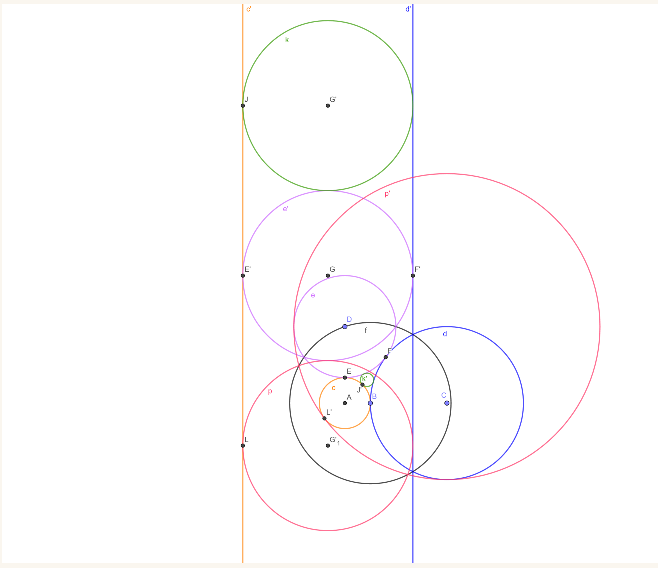
where R'_0 is the radius of the inverted circle of C_0.

Problem 2: three mutually tangent circles

Three circles of radii $1, 2, 3$ are mutually tangent. Two further circles, one circumscribing and one inscribed among the three, are drawn. Find the distance between their centres.

Solution

Place the centres of the three circles at $A(-1, 0)$, $D(-1, 3)$, and $C(3, 0)$.



Choose the circle of inversion f to pass through D and be centred at $B(0, 0)$ (the point of tangency of circles c and d). Its radius is $\sqrt{10}$.

Circles c and d invert to parallel lines c' and d', and e inverts to e'. The inverted circles k and p of k' and p', internally and externally tangent to c, d, e, are tangent to lines c' and d' and to circle e'.

If the centre of the circle of inversion is at (h, k) and the radius is r, then (x, y) inverts to $\left(h + \frac{(x-h)r^2}{(x-h)^2+(y-k)^2},\ k + \frac{(y-k)r^2}{(x-h)^2+(y-k)^2}\right)$.

The point of tangency of c and e, $E(-1, 1)$, inverts to $E'(-5, 5)$. F divides CD in the ratio $2 : 3$, giving $F = (3/5, 9/5)$ which inverts to $F'(5/3, 5)$. The radius of e' is half $|E'F'| = 10/3$.

The tangency point of k and c' at $J(-5, 35/3)$ inverts to $J'(-9/29, 21/29)$, the tangency of k' and c. The coordinates of G' are $(-5/3, 35/3)$. The centre of k' lies on lines BG' and AJ'. The line AJ' is $y/(21/29) = (x + 1)/(20/29)$ and BG' is $y/(35/3) = x/(-5/3)$. Solving simultaneously, the centre of k' is $(-3/23,\ 21/23)$.

Similarly, $L(-5, -5/3)$, the tangency of p and c', inverts to $L'(-9/5,\ -3/5)$. With $G_1' = (-5/3,\ -5/3)$, the centre of p' lies on BG_1' and AL'; solving gives $(3, 3)$.

The distance between the centres of k' and p' is

$$\sqrt{\left(3 + \tfrac{3}{23}\right)^2 + \left(3 - \tfrac{21}{23}\right)^2} \;=\; \frac{24\sqrt{13}}{23}.$$