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Running Total of a Die

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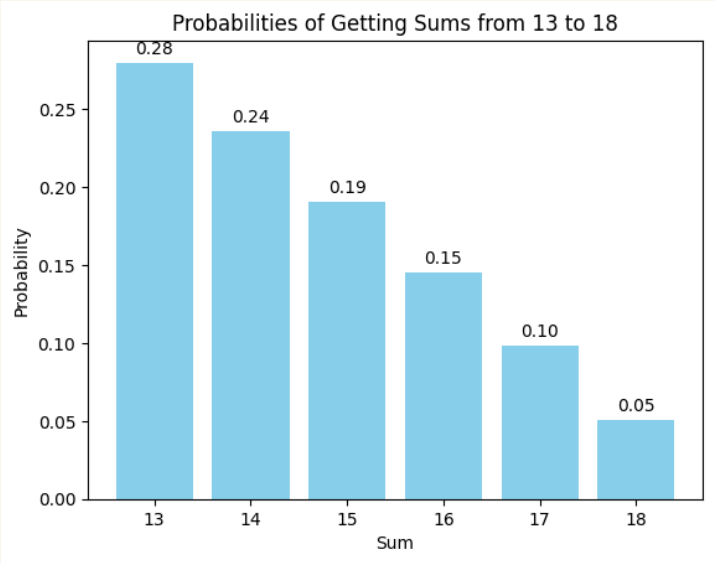
Problem

An ordinary die is rolled until the running total of the rolls first exceeds 12. What is the most likely final total that will be obtained?

Solution 1: Monte Carlo

A straightforward simulation estimates the probabilities of final totals 13 through 18. The most likely final total is **13**, with probability approximately 28%.




```
import numpy as np
import matplotlib.pyplot as plt

def monte_carlo_simulation(n=100_000):
    results = []
    for _ in range(n):
        total = 0
        while total <= 12:
```

```
            total += np.random.randint(1, 7)
        results.append(total)
    return results

def calculate_probabilities(results):
    return {i: results.count(i) / len(results) for i in range(13, 19)}

def plot_probabilities(probabilities):
    fig, ax = plt.subplots()
    bars = ax.bar(probabilities.keys(), probabilities.values(), color='skyblue')
    ax.set_xlabel('Sum')
    ax.set_ylabel('Probability')
    ax.set_title('Probabilities of Final Totals from 13 to 18')
    for bar in bars:
        h = bar.get_height()
        ax.annotate(f'{h:.2f}',
                    xy=(bar.get_x() + bar.get_width() / 2, h),
                    xytext=(0, 3), textcoords="offset points",
                    ha='center', va='bottom')
    plt.show()

results = monte_carlo_simulation(100_000)
probabilities = calculate_probabilities(results)
plot_probabilities(probabilities)
```

Solution 2: integer partitions

The second approach is more interesting. We count the number of ways of arriving at a total between 13 and 18 such that every number in the sum is at most 6 and the running total up to the second-to-last number is at most 12. For example, $8+4+2$ is a valid way to reach 14 but $8+5+1$ is not, since $8+5$ already exceeds 12.

Write $p(n,k)$ for the number of partitions of n in which each part is at most k. The number of ways to reach each target is

Total	Count
13	$p(13,6)$
14	$p(8,6)+p(9,6)+p(10,6)+p(11,6)+p(12,6)$
15	$p(9,6)+p(10,6)+p(11,6)+p(12,6)$
16	$p(10,6)+p(11,6)+p(12,6)$
17	$p(11,6)+p(12,6)$
18	$p(12,6)$

The probability of reaching 14 exceeds that of reaching $15,16,17,18$. To decide between 13 and 14, compare $p(13,6)$ with $\sum_{i=8}^{12} p(i,6)$. The code below gives $\Pr[\text{total}=13]\approx 28\%$ and $\Pr[\text{total}=14]\approx 24\%$, confirming 13 is the most likely final total.

```
from collections import Counter
from math import factorial

def generate_partitions(number, max_part=6):
    def recurse(target, mx, current):
        if target == 0:
            yield list(current)
        else:
```

```
            start = min(mx, target)
            for p in range(start, 0, -1):
                yield from recurse(target - p, p, current + [p])
    return list(recurse(number, max_part, []))

def calculate_probability(number):
    def partition_probability(partition):
        counter = Counter(partition)
        l = len(partition)
        num = factorial(l)
        for _, cnt in counter.items():
            num *= 1 / factorial(cnt)
        return num / 6 ** l
    partitions = generate_partitions(number)
    return round(sum(partition_probability(p) for p in partitions), 2)

print("Total probability for 13:", calculate_probability(13))
prob = sum(calculate_probability(n) / 6 for n in range(8, 13))
print("Total probability for 14:", prob)
```