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# *Charlotte's New Web*

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A puzzle by Xavier Durawa for the week of 5/3.

## *Puzzle*

Charlotte is building a web inside a circular frame. She picks a starting point on the boundary uniformly at random, picks a direction pointing into the disk uniformly at random, walks in a straight line, and lays down a strand of silk. When she reaches the boundary she reflects elastically, in exactly the way a billiard ball or a beam of light would, and lays down the next strand. New strands cross many of the earlier ones. On average, how many of the previous strands does the $n$-th strand cross?

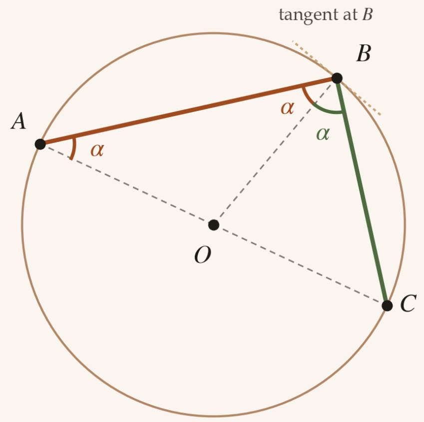
That is the whole question. Getting to a clean answer is a sequence of small moves: a piece of reflection geometry to collapse the two random choices into one, a one-line crossing criterion phrased in terms of distance to the nearest integer, an antipode symmetry, a piecewise integration, and a telescoping sum. The reward at the end is an exact closed form, short enough to fit on one line.

We work in the unit disk $\{(x, y) : x^2 + y^2 \leq 1\}$.

## *Step 1: every chord has the same length*

Charlotte's first chord goes $A \to B$; she reflects and continues $B \to C$. Let $O$ be the centre. Triangle $OAB$ is isosceles $(OA = OB = 1)$, so its base angles at $A$ and $B$ agree: $\angle OAB = \angle OBA = \alpha$.

The radius $OB$ is perpendicular to the tangent at $B$. Reflection preserves the angle to the tangent, hence to the radius $OB$ on the other side, so $\angle OBC = \alpha$ as well. Both triangles $OAB$ and $OBC$ have two unit radii and base angles $\alpha$, so their apex angles at $O$ are both $180° - 2\alpha$; hence the triangles are congruent (SAS), and $|BC| = |AB|$.




So every chord has the same length, and equivalently the same central angle. Call this constant central angle $\delta$. The bounce points $P_0, P_1, P_2, \ldots$ form an arithmetic progression of angles around the circle: $P_k$ sits at angle $\theta_0 + k\delta$.

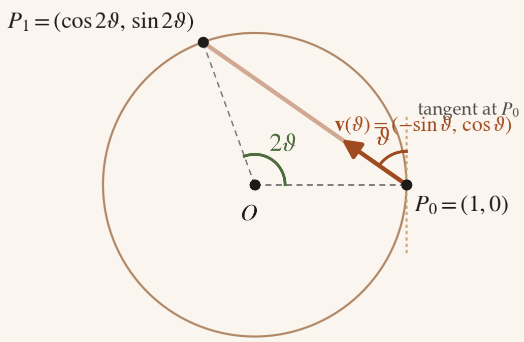
### *Step 2: from two random choices to one*

Place $P_0$ at $(1, 0)$ (rotational symmetry; the puzzle is invariant under rotation of the disk). Parametrise Charlotte's inward direction at $(1, 0)$ by the angle $\vartheta \in (0, \pi)$ from the upward tangent, giving the unit vector $\mathbf{v}(\vartheta) = (-\sin\vartheta, \cos\vartheta)$. The puzzle specifies the direction as uniform, so $\vartheta$ is uniform on $(0, \pi)$. (No Bertrand-style ambiguity here: the random ingredient is the direction at a fixed boundary point, not the chord.)

Walking along $P(t) = (1, 0) + t\,\mathbf{v}(\vartheta)$, the next boundary hit satisfies $\|P(t)\|^2 = 1 - 2t\sin\vartheta + t^2 = 1$, so $t = 2\sin\vartheta$, and

$$P_1 \;=\; (1 - 2\sin^2\vartheta,\; 2\sin\vartheta\cos\vartheta) \;=\; (\cos 2\vartheta,\; \sin 2\vartheta).$$

That places $P_1$ on the circle at angle $2\vartheta$, so $\delta = 2\vartheta$ is uniform on $(0, 2\pi)$.



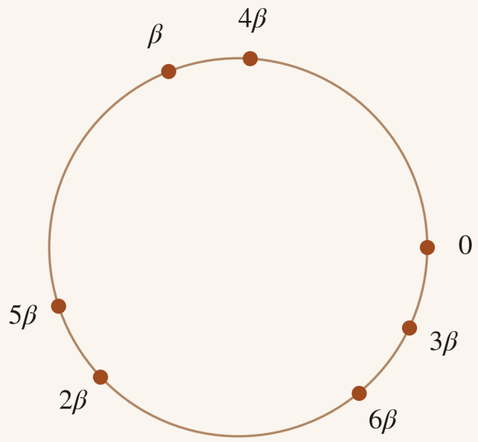
Set

$$\beta := \frac{\delta}{2\pi},$$

so $\beta$ is uniform on $(0, 1)$, and the whole web is determined by this single random parameter. Measuring boundary positions as fractions of the circumference (the boundary becomes $[0, 1)$ with endpoints glued), the bounce points sit at

$$P_0, P_1, P_2, \ldots \quad \text{at positions} \quad 0, \beta, 2\beta, 3\beta, \ldots \quad (\text{mod } 1).$$

The $k$-th strand is the chord joining $P_{k-1}$ to $P_k$, i.e. the chord whose boundary endpoints are $(k-1)\beta$ and $k\beta$ modulo 1.



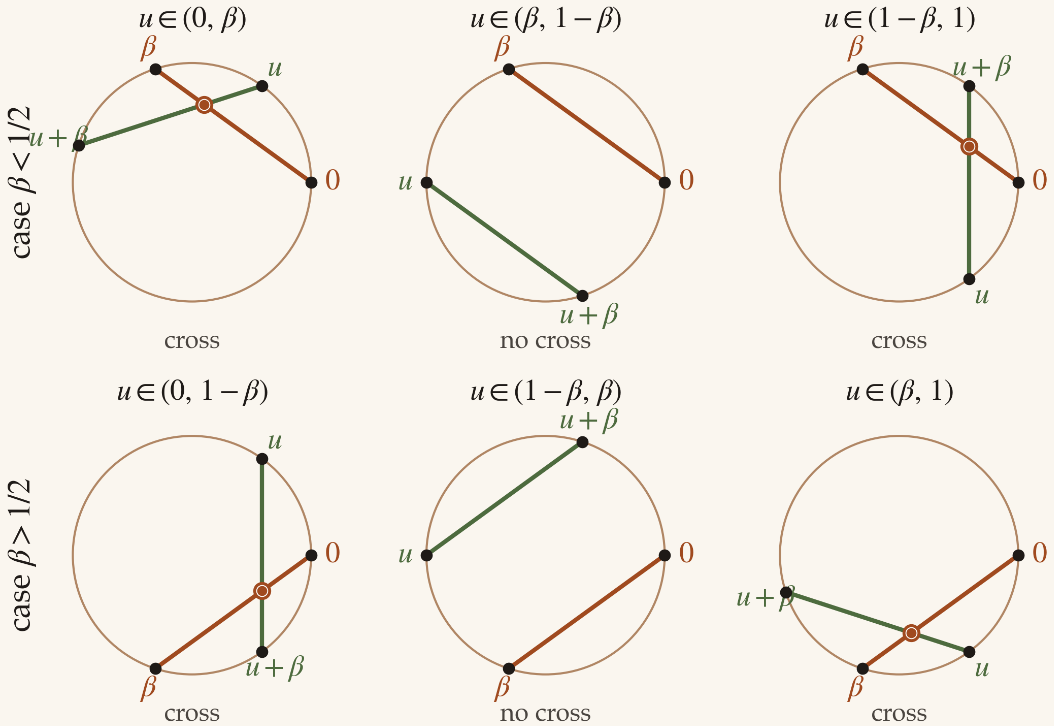

positions $k\beta$ (mod 1) for $k = 0, 1, \ldots, 6$

## *Step 3: when does chord n cross chord k?*

Two chords cross strictly inside the disk iff their four endpoints alternate around the boundary. Alternation is invariant under rotating the boundary, so shift chord $k$ to sit at $\{0, \beta\}$. Chord $n$ then sits at $\{u, u + \beta\}$ mod 1, where

$$u := (n-k)\beta \bmod 1.$$

The chord $\{0, \beta\}$ cuts the boundary into two arcs: $A = (0, \beta)$ of length $\beta$, and $B = (\beta, 1)$ of length $1 - \beta$. Chords cross iff exactly one of $\{u, u + \beta \bmod 1\}$ lies in each arc.




Case $\beta < 1/2$ (top row). Arc $A$ has length $\beta$ and is the shorter one. Three sub-cases for $u$:

- $u \in (0, \beta)$: $u \in A$ and $u + \beta \in (\beta, 2\beta) \subset B$. *Cross.*
- $u \in (\beta, 1-\beta)$: $u \in B$ and $u + \beta \in (2\beta, 1) \subset B$. *No cross.*
- $u \in (1-\beta, 1)$: $u \in B$ and $u + \beta - 1 \in (0, \beta) = A$. *Cross.*

The cross region is $u \in (0, \beta) \cup (1-\beta, 1)$.

Case $\beta > 1/2$ (bottom row). Now arc $B$ has length $1-\beta$ and is the shorter one. Three sub-cases:

- $u \in (0, 1-\beta)$: $u \in A$ and $u + \beta \in (\beta, 1) = B$. *Cross.*
- $u \in (1-\beta, \beta)$: $u \in A$ and $u + \beta - 1 \in (0, 2\beta - 1) \subset A$. *No cross.*
- $u \in (\beta, 1)$: $u \in B$ and $u + \beta - 1 \in (2\beta - 1, \beta) \subset A$. *Cross.*

The cross region is $u \in (0, 1-\beta) \cup (\beta, 1)$.

Both cases together. In each case the cross region is the set of $u$ whose distance to the nearer of $\{0, 1\}$ is less than the length of the shorter arc. Writing the distance from $u$ to its nearer endpoint as $\min(u, 1-u)$, and the shorter-arc length as $\min(\beta, 1-\beta)$, the two cases collapse into one inequality:

$$\boxed{\text{chord } n \text{ crosses chord } k \iff \min(u, 1-u) < \min(\beta, 1-\beta).}$$

The disk has vanished. The puzzle is now: how often is the fractional part of $(n-k)\beta$ closer to $\{0, 1\}$ than $\beta$ itself?

### *Step 4: expectation as a sum of indicator probabilities*

The condition $\min(x, 1-x) < \min(\beta, 1-\beta)$ has a tidier spelling. For any real $y$, write

$$\|y\| \;:=\; \min_{m \in \mathbb{Z}} |y - m|$$

for the distance from $y$ to the nearest integer; this function has period 1 and equals $\min(\{y\}, 1-\{y\})$ on each unit interval. The criterion of step 3 is then $\|(n-k)\beta\| < \|\beta\|$, with the modulo-1 absorbed into $\|\cdot\|$. Let $I_{nk}$ be the indicator of the event that chord $n$ crosses chord $k$:

$$I_{nk} \;=\; \mathbf{1}\{\|(n-k)\beta\| < \|\beta\|\}.$$

Then $N_n$, the number of previous strands that strand $n$ crosses, is the sum of these indicators:

$$N_n \;=\; \sum_{k=1}^{n-1} I_{nk}.$$

Expectation distributes over sums (no independence required), so

$$\mathbb{E}[N_n] \;=\; \sum_{k=1}^{n-1} \mathbb{E}[I_{nk}] \;=\; \sum_{k=1}^{n-1} \mathbb{P}(\|(n-k)\beta\| < \|\beta\|).$$

The probability inside the sum depends only on the gap $m = n - k$, not on $n$ or $k$ separately. Define

$$P_m \;:=\; \mathbb{P}(\|m\beta\| < \|\beta\|), \qquad \beta \sim \mathrm{Unif}(0,1).$$

Then

$$\mathbb{E}[N_n] \;=\; \sum_{m=1}^{n-1} P_m,$$

and the puzzle has reduced to a single problem: compute $P_m$.

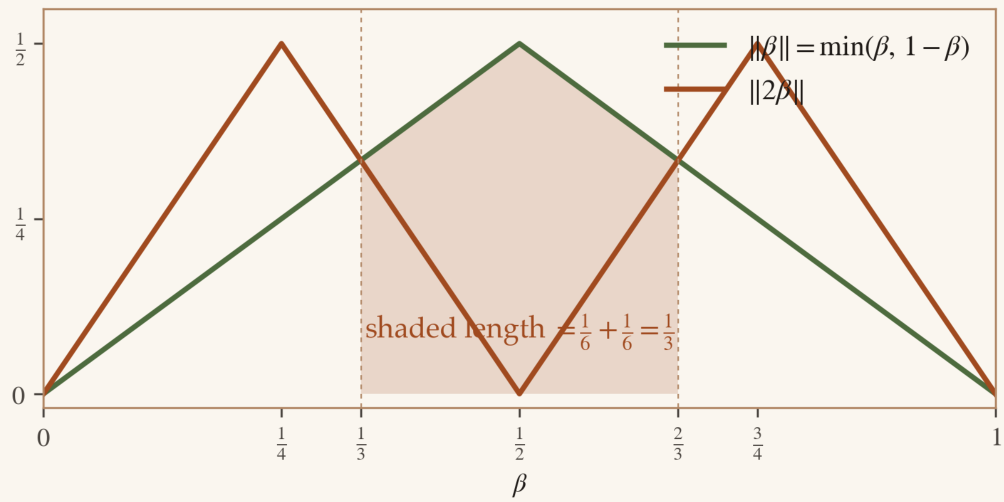
### *Step 5: a worked example, $P_2$*

Let us compute $P_2 = \mathbb{P}(\|2\beta\| < \|\beta\|)$ by hand. The trick is to chop $[0,1]$ into pieces on which both $\|\beta\|$ and $\|2\beta\|$ are linear in $\beta$, and read off the answer piece by piece.

The function $\|\beta\|$ is a tent: it climbs linearly from 0 at $\beta = 0$ up to $1/2$ at $\beta = 1/2$, then descends to 0 at $\beta = 1$. Explicitly $\|\beta\| = \beta$ on $[0, 1/2]$ and $\|\beta\| = 1 - \beta$ on $[1/2, 1]$.

The function $\|2\beta\|$ is the same shape, but at twice the frequency: two tents stacked side by side. A direct computation on each quarter-interval gives

$$\|2\beta\| = \begin{cases} 2\beta & \text{on } [0, \frac{1}{4}], \\ 1 - 2\beta & \text{on } [\frac{1}{4}, \frac{1}{2}], \\ 2\beta - 1 & \text{on } [\frac{1}{2}, \frac{3}{4}], \\ 2 - 2\beta & \text{on } [\frac{3}{4}, 1]. \end{cases}$$




The probability we want is the total length of the $\beta$-interval on which the orange tents (the high-frequency function $\|2\beta\|$) are below the green tent (the low-frequency $\|\beta\|$). Reading off each piece:

$$\begin{aligned}
\beta \in [0, \tfrac{1}{4}]:&\quad 2\beta < \beta \iff \beta < 0. \quad \text{Empty.}\\
\beta \in [\tfrac{1}{4}, \tfrac{1}{2}]:&\quad 1 - 2\beta < \beta \iff \beta > \tfrac{1}{3}. \quad \text{Length } \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}.\\
\beta \in [\tfrac{1}{2}, \tfrac{3}{4}]:&\quad 2\beta - 1 < 1 - \beta \iff \beta < \tfrac{2}{3}. \quad \text{Length } \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{1}{6}.\\
\beta \in [\tfrac{3}{4}, 1]:&\quad 2 - 2\beta < 1 - \beta \iff \beta > 1. \quad \text{Empty.}
\end{aligned}$$

Adding the four contributions: $P_2 = 0 + \frac{1}{6} + \frac{1}{6} + 0 = \frac{1}{3}$.

The same recipe works for any $m$: $\|m\beta\|$ has $2m$ linear pieces, and on each piece the comparison with $\|\beta\|$ is one linear inequality whose solution set is an interval (possibly empty). For larger $m$ the bookkeeping gets out of hand, and we need a closed form. That is the job of the next step.

### *Step 6: a closed form for $P_m$*

The closed form depends on the parity of $m$.

Odd $m \geq 3$: $P_m = 1/2$, by pairing $\beta$ with $\beta + \frac{1}{2}$. Pair each $\beta \in (0, 1)$ with its antipode $\beta' := \beta + \frac{1}{2} \bmod 1$. The pairing partitions $(0, 1)$ into disjoint pairs $\{\beta, \beta'\}$. The borderline case $\|m\beta\| = \|\beta\|$ holds only on finitely many $\beta$, total length zero, so we ignore it. Off this null set, we will show that for $m$ odd exactly one member of each pair satisfies $\|m\beta\| < \|\beta\|$. Half of $(0, 1)$ then satisfies the inequality, giving $P_m = 1/2$.

The function $\|\cdot\|$ traces a tent of height $\frac{1}{2}$ on each unit interval, peaking at the midpoint. Shifting by $\frac{1}{2}$ reflects the tent vertically across the line $y = \frac{1}{4}$:

$$\|x + \tfrac{1}{2}\| = \tfrac{1}{2} - \|x\|.$$

Apply this to $\beta'$ and to $m\beta'$. For the latter, write

$$m\beta' = m\beta + \tfrac{m-1}{2} + \tfrac{1}{2},$$

where $(m-1)/2$ is an integer (this is where odd $m$ matters). Integers slide off $\|\cdot\|$ since it has period 1, leaving

$$\|m\beta'\| \;=\; \|m\beta + \tfrac{1}{2}\| \;=\; \tfrac{1}{2} - \|m\beta\|.$$

The inequality $\|m\beta'\| < \|\beta'\|$ becomes $\frac{1}{2} - \|m\beta\| < \frac{1}{2} - \|\beta\|$, i.e. $\|m\beta\| > \|\beta\|$. Each pair $\{\beta, \beta'\}$ therefore satisfies opposite strict inequalities: one "cross", the other "no cross". Exactly half of $(0, 1)$ gives "cross". □

Even $m = 2k$: $P_{2k} = \dfrac{2k^2 - 1}{4k^2 - 1}$, by direct integration. The pairing argument fails: the offset $(m-1)/2$ stops being an integer, so the half-period shift on $\beta$ no longer flips the sign on $\|m\beta\|$. We have to do the integral.

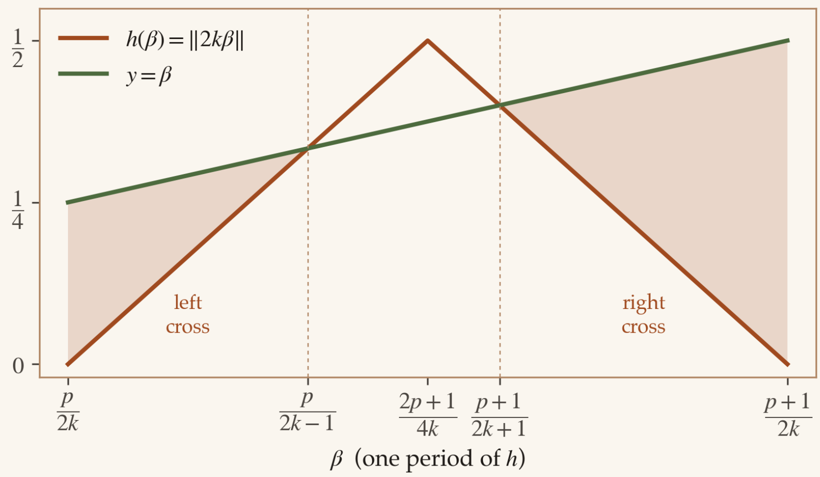
The map $\beta \mapsto 1 - \beta$ leaves both $\|\beta\|$ and $\|2k\beta\|$ unchanged, so the integrand is symmetric about $\beta = 1/2$:

$$P_{2k} \;=\; 2\int_0^{1/2} \mathbf{1}\{\|2k\beta\| < \beta\}\, d\beta,$$

using $\|\beta\| = \beta$ on $[0, 1/2]$.

Write $h(\beta) := \|2k\beta\|$. It has period $1/(2k)$, so $[0, 1/2]$ holds exactly $k$ full periods, indexed by $p = 0, 1, \ldots, k-1$. On the $p$-th period $[\frac{p}{2k}, \frac{p+1}{2k}]$, $h$ is a tent: it climbs from 0 to 1/2 on the left half and falls back to 0 on the right.

$$h(\beta) = \begin{cases} 2k\beta - p & \text{on the left half } [\frac{p}{2k}, \frac{2p+1}{4k}], \\ (p+1) - 2k\beta & \text{on the right half } [\frac{2p+1}{4k}, \frac{p+1}{2k}]. \end{cases}$$




The diagonal $y = \beta$ has slope 1, while the tent has slope $\pm 2k$. The tent crosses $y = \beta$ once on each half, so the cross region inside the period is two short intervals at the ends.

On the left half, $h(\beta) < \beta$ becomes $2k\beta - p < \beta$, i.e. $\beta < p/(2k-1)$. The cut sits inside the half (since $p/(2k-1) \le (2p+1)/(4k)$ reduces to $2k - 2p - 1 \ge 0$, true for $p \le k-1$), so the contributing length is

$$\frac{p}{2k-1} - \frac{p}{2k} = \frac{p}{(2k-1)(2k)}.$$

On the right half, $h(\beta) < \beta$ becomes $\beta > (p+1)/(2k+1)$, contributing length

$$\frac{p+1}{2k} - \frac{p+1}{2k+1} = \frac{p+1}{(2k)(2k+1)}.$$

Sum over $p = 0, 1, \ldots, k-1$, using $\sum p = k(k-1)/2$ and $\sum (p+1) = k(k+1)/2$:

$$\begin{aligned}\int_0^{1/2} \mathbf{1}\{h < \beta\}\, d\beta &= \frac{k(k-1)}{2(2k-1)(2k)} + \frac{k(k+1)}{2(2k)(2k+1)} \\ &= \frac{k-1}{4(2k-1)} + \frac{k+1}{4(2k+1)}.\end{aligned}$$

Double, and combine over $2(2k-1)(2k+1) = 2(4k^2-1)$:

$$P_{2k} = \frac{(k-1)(2k+1) + (k+1)(2k-1)}{2(4k^2-1)} = \frac{4k^2-2}{2(4k^2-1)} = \frac{2k^2-1}{4k^2-1}. \quad \square$$

What the formula is telling us. Rewriting $P_{2k} = \frac{1}{2} - \frac{1}{2(4k^2-1)}$ shows the even-$m$ values are just slightly below 1/2, by an amount $1/(2(4k^2-1))$ that shrinks like $1/k^2$. The odd $P_m$ are exactly 1/2. So the crossing probability $P_m$ tends to 1/2 as the index gap $m$ grows: the larger the gap, the more "uncorrelated" the two strands look, and the closer the crossing chance is to a fair coin. The small downward correction on even $m$ is the residue left over from the geometry; it will be the engine that produces the $1/(4n_*)$ term in the final answer.

### *Step 7: summing the $P_m$'s*

We now want to compute $\mathbb{E}[N_n] = \sum_{m=1}^{n-1} P_m$. Split the sum by the parity of $m$. Among the values $m = 1, 2, \ldots, n-1$:

- $m = 1$ contributes 0.
- The odd values $m = 3, 5, \ldots$ each contribute 1/2. There are $\lfloor n/2 \rfloor - 1$ of them (the odd numbers strictly between 1 and $n$).
- The even values $m = 2, 4, \ldots, 2K$ each contribute $P_{2k} = (2k^2-1)/(4k^2-1)$. Here $K = \lfloor (n-1)/2 \rfloor$, the number of even values among $\{1, 2, \ldots, n-1\}$.

So

$$\mathbb{E}[N_n] = 0 + (\lfloor n/2 \rfloor - 1) \cdot \tfrac{1}{2} + \sum_{k=1}^{K} \frac{2k^2-1}{4k^2-1}.$$

The even-$m$ summand has a tidy rewrite:

$$\frac{2k^2-1}{4k^2-1} = \frac{1}{2} - \frac{1}{2(4k^2-1)}.$$

Substituting,

$$\mathbb{E}[N_n] \;=\; (\lfloor n/2\rfloor - 1)\cdot \tfrac{1}{2} + \tfrac{K}{2} \;-\; \frac{1}{2}\sum_{k=1}^{K}\frac{1}{4k^2-1}.$$

The first two terms count the odd-$m$ and even-$m$ contributors respectively (each at weight $1/2$). Together they cover every $m$ from 2 to $n-1$, totalling $n-2$ contributors. (For $n=5$: one odd value $m=3$ and two even values $m=2,4$, total three.) So those two terms add to $(n-2)/2$, and

$$\mathbb{E}[N_n] \;=\; \frac{n-2}{2} \;-\; \frac{1}{2}\sum_{k=1}^{K}\frac{1}{4k^2-1}, \qquad K = \lfloor (n-1)/2\rfloor.$$

The only mystery left is the tail sum.

## *Step 8: telescope the sum*

The denominator factors:

$$4k^2 - 1 \;=\; (2k-1)(2k+1).$$

Partial fractions then give

$$\frac{1}{(2k-1)(2k+1)} \;=\; \frac{1}{2}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right).$$

The sum now telescopes: each negative term in one summand cancels the positive term in the next. Writing the first few out:

$$\begin{aligned}
\sum_{k=1}^{K}\frac{1}{4k^2-1} &= \frac{1}{2}\sum_{k=1}^{K}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)\\
&= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2K-1}-\frac{1}{2K+1}\right)\right]\\
&= \frac{1}{2}\left(1-\frac{1}{2K+1}\right)\\
&= \frac{1}{2}-\frac{1}{2(2K+1)}.
\end{aligned}$$

The middle line is where the work happens: $-1/3$ from the first parenthesis kills the $+1/3$ from the second, $-1/5$ kills $+1/5$, and so on down the line. Only the very first $+1$ and the very last $-1/(2K+1)$ survive.

Substituting back into Step 7,

$$\mathbb{E}[N_n] \;=\; \frac{n-2}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2(2K+1)}\right) \;=\; \frac{n-2}{2} - \frac{1}{4} + \frac{1}{4(2K+1)} \;=\; \frac{2n-5}{4} + \frac{1}{4(2K+1)}.$$

## *Step 9: the exact closed form*

The final job is to simplify $2K+1$ in terms of $n$. Recall $K = \lfloor (n-1)/2\rfloor$:

- If $n$ is odd, $n-1$ is even, $K = (n-1)/2$, so $2K+1 = n$.

- If $n$ is even, $n-1$ is odd, $K = (n-2)/2$, so $2K+1 = n-1$.

Define $n_* = n$ for $n$ odd, and $n_* = n-1$ for $n$ even. We have proved

$$\boxed{\mathbb{E}[N_n] = \frac{2n-5}{4} + \frac{1}{4\,n_*}.}$$

That is the answer. Two things are worth noticing.

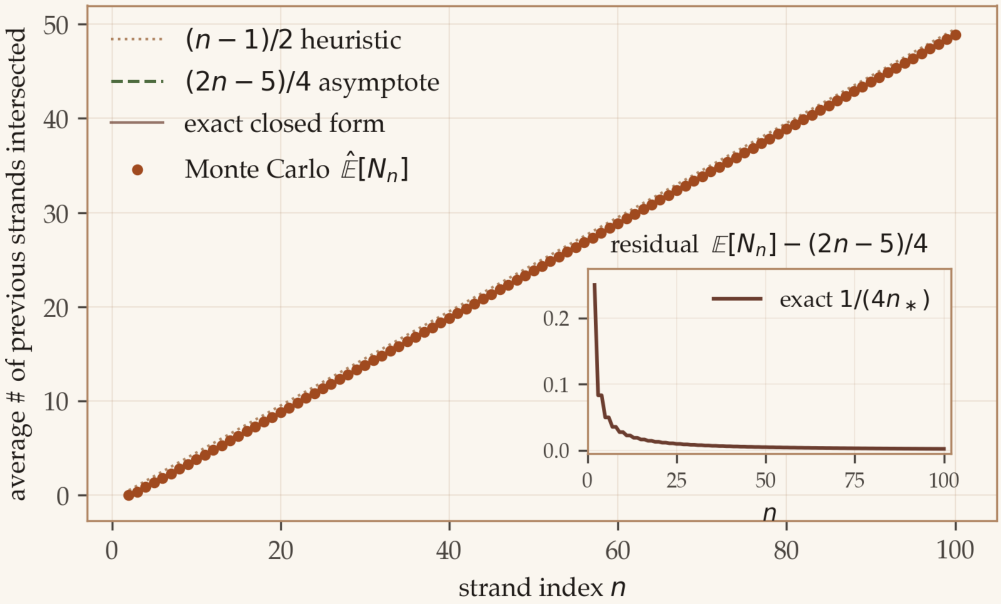
*First*, as $n \to \infty$ the residual $1/(4n_*)$ vanishes and $\mathbb{E}[N_n] \to (2n-5)/4 = \frac{n}{2} - \frac{5}{4}$. A plausible-looking guess might be that the $n$-th strand crosses "half of the previous ones," i.e. $(n-1)/2 = \frac{n}{2} - \frac{1}{2}$. The true answer undershoots that guess by exactly $\frac{3}{4}$ in the limit. The discrepancy comes from the fact that consecutive strands ($m = 1$) share a boundary endpoint and so never cross at all, which systematically depresses the count.

*Second*, the residual $1/(4n_*)$ decays like $1/(4n)$ but does so in pairs of equal height: $n = 3$ and $n = 4$ both give residual $1/12$; $n = 5$ and $n = 6$ both give $1/20$; and so on. Going from an odd $n$ to the next $n+1$ does not move $n_*$ at all, so the residual stays put.

### *Small-n table*

| $n$ | $\mathbb{E}[N_n]$ exact | decimal | $(2n-5)/4$ |
|---|---|---|---|
| 3 | 1/3 | 0.3333 | 0.2500 |
| 4 | 5/6 | 0.8333 | 0.7500 |
| 5 | 13/10 | 1.3000 | 1.2500 |
| 10 | 34/9 | 3.7778 | 3.7500 |
| 20 | 333/38 | 8.7632 | 8.7500 |
| 30 | 399/29 | 13.7586 | 13.7500 |
| 50 | 1164/49 | 23.7551 | 23.7500 |
| 100 | 9653/198 | 48.7525 | 48.7500 |

## *Monte Carlo verification*




Each dot is the average of 80,000 random $\beta$ draws. For each draw we build the boundary positions $\{0, \beta, 2\beta, \ldots, 100\beta\}$ mod 1 and, for each strand $n$, count by direct geometry how many of the previous $n-1$ strands it crosses. The dots sit on the closed-form curve all the way out to $n = 100$. The inset shows the exact residual $\mathbb{E}[N_n] - (2n-5)/4 = 1/(4n_*)$, which is already about 0.0025 at $n = 100$ and decays as $1/(4n)$ in pairs of equal height.

## *Looking back at the moves*

Three turns took us from a billiard-in-a-circle to the closed form, and they are worth naming.

*The first move was geometric.* The reflection law plus a pair of isosceles triangles forced every chord to subtend the same central angle. Two random choices (start point and direction) collapsed into one ($\delta$, equivalently $\beta$). Once the chord length is fixed, the puzzle is no longer about a billiard at all; it is about an arithmetic progression $\beta, 2\beta, 3\beta, \ldots$ on a circle.

*The second move was combinatorial.* The crossing criterion threw away the indices $i$ and $j$ entirely and reduced the question to whether $\|m\beta\| < \|\beta\|$ for the gap $m$. The expected number of crossings became a sum of probabilities $P_m$ indexed by gap size, one $P_m$ per row of the eventual answer.

*The third move was symmetry, twice.* An antipode shift $\beta \mapsto \beta + \frac{1}{2}$ killed every odd $P_m$ at exactly $1/2$, and a piecewise integration handled the even $P_m$. The remaining sum then telescoped.

The closed form $\mathbb{E}[N_n] = (2n-5)/4 + 1/(4n_*)$ is a record of all three moves: the leading $(2n-5)/4$ is what you would get if every $P_m$ were exactly $1/2$ (the antipode-symmetric value), shifted down by $3/4$ to account for the boundary effect that strand 1 never crosses ($P_1 = 0$); the residual $1/(4n_*)$ is the tail of the small downward corrections to even $P_{2k}$, telescoped.

## *Python code*

```python
import numpy as np
from fractions import Fraction

def positions(n, beta):
    """Boundary positions of P_0, P_1, ..., P_n in [0, 1)."""
    return (np.arange(n + 1) * beta) % 1.0

def to_xy(pos):
    """Map a position in [0, 1) to its (x, y) on the unit circle."""
    a = 2 * np.pi * pos
    return np.column_stack([np.cos(a), np.sin(a)])

def segments_cross(p1, p2, p3, p4, eps=1e-12):
    """Open segments p1-p2 and p3-p4 cross strictly inside the disk?"""
    d1, d2 = p2 - p1, p4 - p3
    denom = d1[0]*d2[1] - d1[1]*d2[0]
    if abs(denom) < eps:
        return False
    diff = p3 - p1
    t = (diff[0]*d2[1] - diff[1]*d2[0]) / denom
    s = (diff[0]*d1[1] - diff[1]*d1[0]) / denom
    return eps < t < 1-eps and eps < s < 1-eps

def E_Nn_montecarlo(n_max=100, n_trials=80_000, seed=0):
    """Estimate E[N_n] for n = 2..n_max by Monte Carlo."""
    rng = np.random.default_rng(seed)
    counts = np.zeros(n_max + 1)
    for _ in range(n_trials):
        beta = rng.uniform(0, 1)
        pts = to_xy(positions(n_max, beta))
        for n in range(2, n_max + 1):
            a, b = pts[n-1], pts[n]
            counts[n] += sum(
                segments_cross(a, b, pts[k-1], pts[k])
                for k in range(1, n)
            )
    return counts / n_trials

def exact_E_Nn(n):
    """Closed form: (2n-5)/4 + 1/(4 n_*) where
    n_* = n if n is odd, else n - 1."""
    n_star = n if n % 2 == 1 else n - 1
    return Fraction(2*n - 5, 4) + Fraction(1, 4 * n_star)
```