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The Apollonian Gasket

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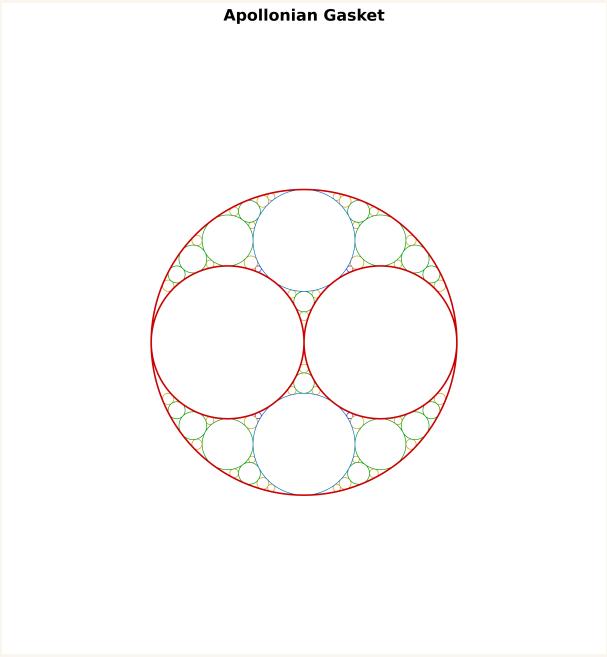
April 2025

A fractal of mutually tangent circles, generated from Descartes' Circle Theorem and its complex extension.

Introduction

The *Apollonian gasket* is a fractal generated by starting with three mutually tangent circles and recursively filling the curvilinear triangular gaps with new circles tangent to all three of their neighbours. Iterating produces a nested pattern of self-similar tangencies whose Hausdorff dimension is approximately 1.3057.

The construction rests on Descartes' Circle Theorem (1643) and its complex extension due to Lagarias, Mallows, and Wilks (2002). The first determines the radius of the fourth circle; the second determines its centre.



Descartes' Circle Theorem

For each circle, define the *signed curvature*

$$k = \begin{cases} +1/r & \text{if the centre lies on the exterior of the configuration,} \\ -1/r & \text{if the circle contains the others.} \end{cases}$$

Theorem (Descartes, 1643)**.** For four mutually tangent circles with signed curvatures k_1, k_2, k_3, k_4,

$$(k_1 + k_2 + k_3 + k_4)^2 \;=\; 2\,(k_1^2 + k_2^2 + k_3^2 + k_4^2).$$

Solving for k_4 given k_1, k_2, k_3:

$$k_4 \;=\; k_1 + k_2 + k_3 \;\pm\; 2\sqrt{k_1k_2 + k_1k_3 + k_2k_3}.$$

The two signs give the two circles tangent to all three given circles (one fitting in the inner curvilinear triangle, the other fitting outside it).

An elementary proof, after Levrie (2019)

Connect the centres of the four circles to form four sub-triangles. The key geometric observation is that the area of the largest sub-triangle equals the sum of the areas of the other three.

For mutually tangent circles, the distance between centres i and j is $r_i + r_j$. Heron's formula on each sub-triangle gives

$$\sqrt{r_1r_2r_3(r_1 + r_2 + r_3)} \;=\; \sum_{\text{three smaller triangles}} \sqrt{r_ar_br_c(r_a + r_b + r_c)},$$

where the smaller-triangle terms are the three permutations missing one of the circles. (When one circle contains the others, take its radius as negative; the equation still holds.)

Let $s = r_1 + r_2 + r_3 + r_4$, $\alpha = \sum 1/r_i$, $\beta = \sum 1/r_i^2$. Dividing through by $\sqrt{s}$ and squaring twice (carefully, isolating cross-terms each time) collapses the equation to

$$(2\beta - \alpha^2)(2\beta - \alpha^2 + 8\alpha/s) = 0.$$

The second factor must be non-zero (else the equation that produced it would be negative), so $2\beta = \alpha^2$, which is Descartes' theorem.

The complex extension

Descartes' theorem fixes the radius of the fourth circle but leaves its centre under-determined. The complex extension cures this by promoting the centre z from a real coordinate pair to a complex number, then defining the *complex curvature*

$$\zeta \;=\; k\,z \quad (z \in \mathbb{C}).$$

Theorem (Complex Descartes, Lagarias–Mallows–Wilks, 2002)**.** For four mutually tangent circles with signed curvatures k_j and complex centres z_j,

$$(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4)^2 \;=\; 2\,(\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2).$$

The algebraic form is identical to the real Descartes equation; the variables are complex. Solving for ζ_4:

$$\zeta_4 \;=\; \zeta_1 + \zeta_2 + \zeta_3 \;\pm\; 2\sqrt{\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_2\zeta_3}.$$

The square root is now a complex square root, which itself has two branches. Combined with the $\pm$ in k_4, this gives four candidate circles per triple; geometric tangency picks out the two valid ones.

The new circle's centre is recovered by

$$z_4 \;=\; \zeta_4 \,/\, k_4.$$

Sketch of the derivation

For two externally tangent circles, $|z_i - z_j| = 1/k_i + 1/k_j$. Multiplying by $k_i k_j$ on both sides puts the tangency condition into a form that involves the complex curvatures $\zeta_i = k_i z_i$ rather than the centres alone. Squaring and writing $|z_i - z_j|^2 = (z_i - z_j)\overline{(z_i - z_j)}$ gives, for each pair,

$$k_i^2 k_j^2 \,(|z_i|^2 + |z_j|^2 - z_i\overline{z_j} - \overline{z_i} z_j) \;=\; (k_i + k_j)^2.$$

Summing the six tangency conditions (one per pair of circles) and rearranging with the identity $\sum_{i\neq j} \zeta_i\overline{\zeta_j} = (\sum \zeta_i)(\overline{\sum \zeta_j}) - \sum |\zeta_i|^2$ produces the quadratic identity above. The same algebra that gave the real Descartes theorem now gives the complex one, because the manipulations never used reality of the variables.

Generating the gasket

A standard initial configuration is two unit circles externally tangent to each other and a containing circle of radius 2:

- C_1: centre $-\frac{1}{2}$, $k_1 = 2$
- C_2: centre $+\frac{1}{2}$, $k_2 = 2$
- C_3: centre 0, $k_3 = -1$ (containing circle, negative curvature)

For each triple of mutually tangent circles, the complex Descartes formula yields up to two new tangent circles, each filling one of the curvilinear triangular gaps. Each new circle pairs with two of the originals to form a fresh triple; the recursion fills the gasket.

A queue-based traversal processes each tangent triple exactly once, avoiding the exponential cost of repeatedly checking $\binom{n}{3}$ combinations. Each triple emits at most two valid circles; each valid circle emits three new triples (paired with each pair from the original).

Python implementation

```
import numpy as np
from collections import deque
import matplotlib.pyplot as plt
from matplotlib.patches import Circle as MplCircle
```

```
class Circle:
    def __init__(self, k, z):
        self.k = k              # signed curvature
        self.z = z              # complex centre
        self.r = abs(1.0 / k) if k != 0 else float('inf')


def descartes_curvatures(k1, k2, k3):
    """The two curvatures k4 from real Descartes."""
    s = k1 + k2 + k3
    d = 2 * np.sqrt(k1 * k2 + k1 * k3 + k2 * k3)
    return s + d, s - d


def fourth_circles(c1, c2, c3):
    """Up to four candidate fourth circles; only some are geometrically valid."""
    k_plus, k_minus = descartes_curvatures(c1.k, c2.k, c3.k)
    z1, z2, z3 = c1.k * c1.z, c2.k * c2.z, c3.k * c3.z
    s = z1 + z2 + z3
    root = np.sqrt(z1 * z2 + z1 * z3 + z2 * z3)
    out = []
    for k4 in (k_plus, k_minus):
        for sign in (+1, -1):
            zeta4 = s + sign * 2 * root
            z4 = zeta4 / k4 if k4 != 0 else 0 + 0j
            out.append(Circle(k4, z4))
    return out


def tangent(c1, c2, tol=1e-3):
    d = abs(c1.z - c2.z)
    return min(abs(d - (c1.r + c2.r)), abs(d - abs(c1.r - c2.r))) < tol


def gasket(c1, c2, c3, max_depth=6, min_r=0.005):
    seen = set()
    out = [c1, c2, c3]

    def key(c):
        return (round(c.k, 3), round(c.z.real, 3), round(c.z.imag, 3))

    for c in out:
        seen.add(key(c))

    q = deque([(c1, c2, c3, 0)])
    while q:
        ca, cb, cc, depth = q.popleft()
        if depth >= max_depth:
            continue
        for c4 in fourth_circles(ca, cb, cc):
            if c4.r < min_r or c4.r > 100:
                continue
            if not (tangent(c4, ca) and tangent(c4, cb) and tangent(c4, cc)):
                continue
            k = key(c4)
            if k in seen:
                continue
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            seen.add(k)
            out.append(c4)
            q.extend([
                (c4, ca, cb, depth + 1),
                (c4, ca, cc, depth + 1),
                (c4, cb, cc, depth + 1),
            ])
    return out


# Initial configuration: two unit-disc tangent circles inside a radius-2 container.
c1 = Circle(2, -0.5 + 0j)
c2 = Circle(2, +0.5 + 0j)
c3 = Circle(-1, 0 + 0j)

circles = gasket(c1, c2, c3, max_depth=8, min_r=0.005)

fig, ax = plt.subplots(figsize=(8, 8))
for c in circles:
    ax.add_patch(MplCircle((c.z.real, c.z.imag), c.r,
                           fill=False, lw=0.5))
ax.set_xlim(-1.1, 1.1); ax.set_ylim(-1.1, 1.1)
ax.set_aspect('equal'); ax.axis('off')
plt.show()
```

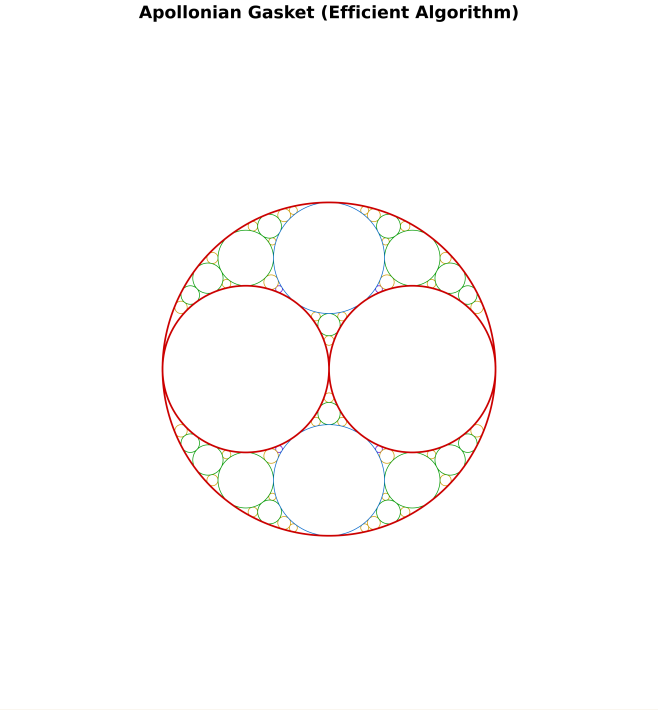

References

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