An experiment in pedagogical temperament

Sheldon Cooper
attends an
Oxbridge Maths Interview

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Three hundred questions, one whiteboard

Vamshi Jandhyala

voice of Dr. Sheldon Cooper

London

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First published in Great Britain in 2026 by Vamshi Jandhyala, London.

A comedic volume: a reconstructed transcript of a theoretical physicist sitting an Oxbridge mathematics interview. The questions are genuine and worked in full; the bedside manner is not. This is an independent work of parody and is not authorised by, affiliated with, or endorsed by any television programme or its rights-holders.

A catalogue record for this book is available from the British Library.

ISBN 978-1-8000-0006-3 (paperback)

Typeset in TeX Gyre Pagella by the author.

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A note to the reader

What follows is a transcript — reconstructed from memory, smoke, and a faintly scorched napkin — of Dr. Sheldon Cooper's attempt to sit an Oxbridge maths entrance interview. The questions are genuine. The answers are not *wrong*, strictly speaking; they are merely delivered by a theoretical physicist who considers pure mathematics a minor applied branch of his own discipline and who cannot resist informing his interviewer of this at every opportunity.

The mathematics, Dr. Cooper assures us, is rigorous. The bedside manner, his colleagues assure us with equal conviction, does not improve with practice.

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Chapter 1

The Interview

The interviewer's office. A small table. One whiteboard. Dr. Cooper is seated, having already informed the interviewer that the whiteboard markers are not expo-grade and that the chair height violates ergonomic specifications published in 1987*.*

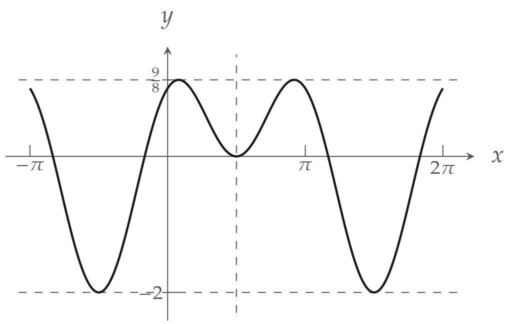
Question 1.1. Interviewer. Sketch $y = \sin x + \cos 2x$ and identify all symmetries.

Sheldon. Use $\cos 2x = 1 - 2\sin^2 x$ and let $u = \sin x$:

$$y = 1 + u - 2u^2 = -2\left(u - \tfrac{1}{4}\right)^2 + \tfrac{9}{8}.$$

A downward parabola in $u \in [-1, 1]$ with maximum $\frac{9}{8}$ at $u = \frac{1}{4}$ (i.e. $x = \arcsin\frac{1}{4}$ and reflections) and minimum -2 at $u = -1$ (i.e. $x = -\frac{\pi}{2} + 2k\pi$). Zeros occur at $u = 1$ and $u = -\frac{1}{2}$.

Symmetries. f is 2π-periodic, and $f(\pi - x) = \sin(\pi - x) + \cos(2\pi - 2x) = f(x)$, so the graph is symmetric about every vertical line $x = \frac{\pi}{2} + k\pi$. Periodicity is the mathematician's preferred way of disguising a translation; one of my least favourite ways.



Question 1.2. Interviewer. Suppose n is an integer. Show that $n^3 - n$ is divisible by 6.

Sheldon. I shall overlook the phrase "suppose n is an integer," which is the mathematician's equivalent of the economist's "suppose a cow is spherical," and proceed. Factor:

$$n^3 - n = (n-1)\,n\,(n+1),$$

the product of three consecutive integers. Among any three consecutive integers exactly one is divisible by 3 and at least one is even, so the product is divisible by $2 \cdot 3 = 6$. I trust you were not expecting me to invoke Fermat's little theorem; that would be akin to using the Large Hadron Collider to open a jar of pickles.

Question 1.3. Interviewer. How many multiples of 2013 have exactly 2013 factors?

Sheldon. Factor $2013 = 3 \cdot 11 \cdot 61$. An integer $N = \prod p_i^{a_i}$ has $\tau(N) = \prod(a_i + 1)$ divisors. For $\tau(N) = 3 \cdot 11 \cdot 61$ and N divisible by $3 \cdot 11 \cdot 61$:

- Fewer than three distinct primes cannot absorb the three primes $3, 11, 61$ required by divisibility.
- More than three distinct primes would force the product $\prod(a_i+1)$ to have more than three factors ≥ 2; impossible

since 2013 has only three prime factors counted with multiplicity.

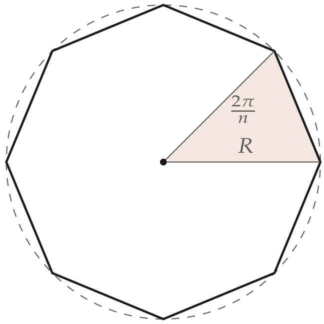
So N has exactly three distinct prime factors with $(a_1+1)(a_2+1)(a_3+1) = 3 \cdot 11 \cdot 61$, i.e. exponents $\{2, 10, 60\}$ in some order, assigned to $\{3, 11, 61\}$. That is $3! = 6$ assignments, so there are exactly 6 such multiples.

Question 1.4. Interviewer. What is the area of an n-sided regular polygon inscribed in a circle of radius R?

Sheldon. Dissect the polygon into n congruent isosceles triangles meeting at the centre, each with two sides of length R and apex angle $\frac{2\pi}{n}$. Each has area $\frac{1}{2}R^2 \sin(2\pi/n)$, so

$$A_n = \frac{nR^2}{2} \sin\left(\frac{2\pi}{n}\right).$$

As $n \to \infty$, $\sin(2\pi/n) \sim 2\pi/n$ and $A_n \to \pi R^2$: Archimedes' method of exhaustion in one line.



Question 1.5. Interviewer. How many trailing zeros does 365! have?

Sheldon. Count factors of 5 (factors of 2 are in ruinous surplus). By Legendre's formula,

$$v_5(365!) = \left\lfloor \frac{365}{5} \right\rfloor + \left\lfloor \frac{365}{25} \right\rfloor + \left\lfloor \frac{365}{125} \right\rfloor = 73 + 14 + 2 = 89.$$

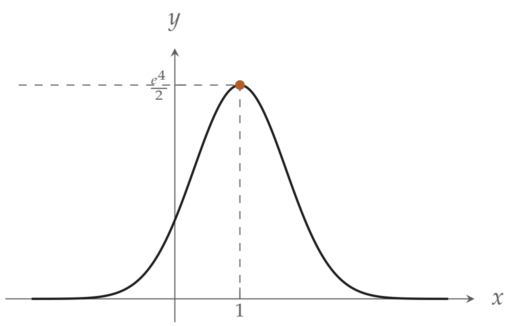
Incidentally, 365 is only the number of days in a tropical year *on average*; the Earth has the audacity to take 365.2422 solar days per orbit, which is why we endure the Gregorian leap-year system, a correction devised by a *physician*, Aloysius Lilius. A physician. For a calendar.

Question 1.6. Interviewer. Sketch $y = \frac{1}{2}e^{-x^2+2x+3}$.

Sheldon. Complete the square: $-x^2 + 2x + 3 = -(x-1)^2 + 4$, so

$$y = \frac{e^4}{2}\, e^{-(x-1)^2}.$$

A Gaussian bell curve centred at $x = 1$ with peak $\frac{e^4}{2} \approx 27.30$, inflection points at $x = 1 \pm \frac{1}{\sqrt{2}}$ (where $y'' = 0$), and reflection symmetry about $x = 1$. As $|x-1| \to \infty$, $y \to 0^+$ strictly.



Question 1.7. Interviewer. Sketch the curve $x^2y^2 = \dfrac{3}{x+y}$.

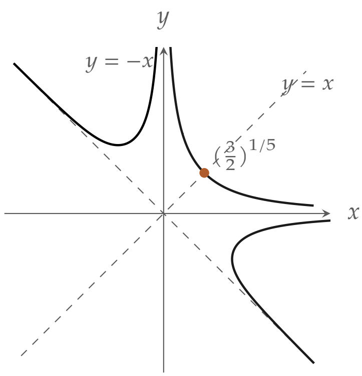
Sheldon. Clear denominators to obtain the polynomial form

$$(xy)^2(x+y) \;=\; 3, \qquad \text{i.e. } x^3y^2 + x^2y^3 = 3.$$

The curve is symmetric under $(x, y) \leftrightarrow (y, x)$, and the right-hand side $3 > 0$ forces $x + y > 0$ together with $xy \neq 0$.

First-quadrant branch. On the diagonal $y = x$ the equation becomes $2x^5 = 3$, giving the central point $((3/2)^{1/5}, (3/2)^{1/5}) \approx (1.08, 1.08)$. As $x \to 0^+$ the term x^3y^2 vanishes faster than x^2y^3 (their ratio is $x/y \to 0$), so the dominant balance is $x^2y^3 \approx 3$, hence $y \sim (3/x^2)^{1/3} \to +\infty$. The branch therefore approaches the positive y-axis (and, by symmetry, the positive x-axis) as a horn going to infinity along the curve $y = (3/x^2)^{1/3}$ rather than as a true linear asymptote.

Second- and fourth-quadrant branches. Here $xy < 0$ but $(xy)^2 > 0$; the constraint $x + y > 0$ permits the region above the line $y = -x$ in the second quadrant and to its right in the fourth. Along $y = -x + \varepsilon$ with $\varepsilon \to 0^+$ and $|x| \to \infty$, the polynomial form gives $(xy)^2\varepsilon = 3$, i.e. $\varepsilon \sim 3/(xy)^2 \to 0^+$, so each off-diagonal branch approaches the line $y = -x$ as a genuine slant asymptote.

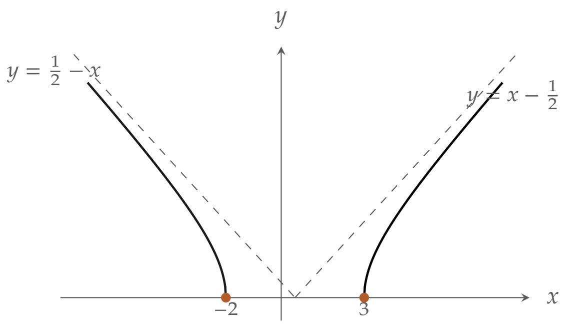


Question 1.8. Interviewer. Sketch $y = \sqrt{x^2 - x - 6}$.

Sheldon. Factor: $x^2 - x - 6 = (x - 3)(x + 2)$, so the domain is $x \le -2$ or $x \ge 3$, and $y \ge 0$. Squaring and completing the square:

$$\left(x - \tfrac{1}{2}\right)^2 - y^2 = \tfrac{25}{4},$$

a rectangular hyperbola centred at $(\frac{1}{2}, 0)$ with semi-axis $\frac{5}{2}$. The graph $y = \sqrt{\cdots}$ is the upper half of the two branches. Asymptotes are $y = x - \frac{1}{2}$ as $x \to +\infty$ and $y = \frac{1}{2} - x$ as $x \to -\infty$. The curve meets the x-axis at -2 and 3.



Question 1.9. Interviewer. What is the remainder of $1!+2!+3!+\cdots+100!$ when divided by 7?

Sheldon. For $n \geq 7$, $n!$ contains a factor of 7 and vanishes modulo 7. The remaining terms:

$$1+2+6+24+120+720 \equiv 1+2+6+3+1+6 \equiv 19 \equiv 5 \pmod{7}.$$

The remainder is 5.

Question 1.10. Interviewer. Which is larger, e^π or π^e?

Sheldon. Take logarithms: comparing e^π and π^e reduces to comparing π and $e \ln \pi$, i.e. to the sign of $f(e) - f(\pi)$ where

$$f(x) = \frac{\ln x}{x}, \qquad f'(x) = \frac{1 - \ln x}{x^2}.$$

f is increasing on $(0, e)$ and decreasing on (e, ∞), with maximum at $x = e$. Since $\pi > e$, $f(e) > f(\pi)$, hence $e^\pi > \pi^e$. Numerically, $e^\pi \approx 23.141$ versus $\pi^e \approx 22.459$.

e^π is Gelfond's constant, proved transcendental by Aleksandr Gelfond in 1934, thereby settling Hilbert's seventh problem. You did not ask. I volunteer it freely.

Question 1.11. Interviewer. Show that the only prime triple $(p, p+2, p+4)$ is $(3, 5, 7)$.

SHELDON. Modulo 3, the residues $\{p, p+2, p+4\} \equiv \{p, p+2, p+1\}$ are a permutation of $\{0, 1, 2\}$, so exactly one term is divisible by 3. For all three to be prime, that term must equal 3 itself. Hence $p = 3$ and the unique triple is $(3, 5, 7)$.

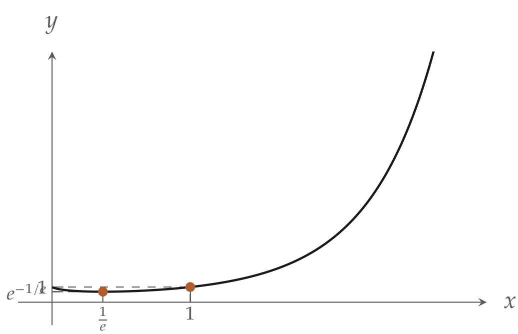
QUESTION 1.12. INTERVIEWER. Differentiate $y = x^x$ and sketch it.

SHELDON. The natural domain is $x > 0$, where $x^x = e^{x \ln x}$ is unambiguously defined; for $x \leq 0$ the expression is either undefined (at $x = 0$ as a limit, though $\lim_{x \to 0^+} x^x = 1$) or multivalued over the reals (negative x with non-integer exponent), so I restrict to $(0, \infty)$.

Take logarithms: $\ln y = x \ln x$. Differentiate implicitly:

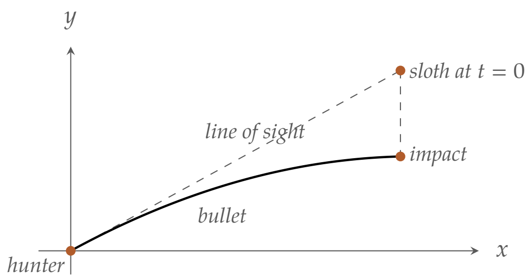
$$\frac{y'}{y} = \ln x + 1 \implies y' = x^x(\ln x + 1).$$

$y' = 0$ at $x = 1/e$, and y' changes sign from negative to positive there, giving the global minimum $y(1/e) = e^{-1/e} \approx 0.692$. As $x \to 0^+$, $x \ln x \to 0$, so $y \to 1$. At $x = 1$, $y = 1$. For $x > 1$, y grows faster than any polynomial.



QUESTION 1.13. INTERVIEWER. A hunter aims at a sloth in a tree. At the instant the hunter fires, the sloth lets go. Will the sloth be hit?

SHELDON. The "Monkey and Hunter" problem. Assume uniform gravitational acceleration g, no air resistance, a point sloth, a point bullet, and a rigid Earth. In the hunter's frame both projectiles experience the same downward acceleration g; the bullet's *relative* trajectory with respect to the sloth is therefore a straight line along the original line of sight. Hence the bullet strikes the sloth at time $t = d/(v\cos\theta)$, where d is the horizontal distance and $v\cos\theta$ the horizontal muzzle velocity. Both have fallen $\frac{1}{2}gt^2$ at the moment of impact. Newton, 1687, effectively.



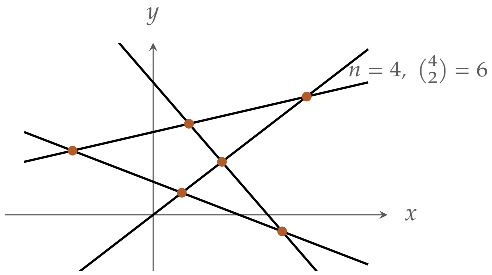
A sloth, incidentally, is a xenarthran, not a primate. My childhood zoologist Bradford was very particular on this point, as is the International Union for Conservation of Nature.

QUESTION 1.14. INTERVIEWER. If you have n non-parallel lines in a plane, how many points of intersection are there?

SHELDON. Each pair of non-parallel lines meets in exactly one point, so at most $\binom{n}{2} = \frac{n(n-1)}{2}$ intersection points exist, with equality iff no three lines are concurrent. If k lines concur at a single point, that point is counted $\binom{k}{2}$ times rather than once, reducing the total; with concurrencies at points of multiplicities $k_1, k_2, \ldots$,

$$\#(\text{intersection points}) = \binom{n}{2} - \sum_i \left(\binom{k_i}{2} - 1\right).$$

In general position, the answer is simply $\frac{n(n-1)}{2}$.

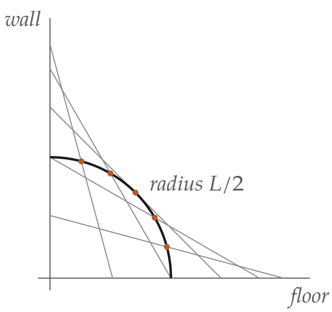


Question 1.15. Interviewer. A ladder of length L leans against a vertical wall with its foot on the ground. As the foot slides out and the ladder falls, what shape does the middle rung trace?

Sheldon. Place the foot at $(a, 0)$ and the top at $(0, b)$; rigidity forces $a^2 + b^2 = L^2$. The midpoint of the ladder is $(\frac{a}{2}, \frac{b}{2})$, whose coordinates satisfy

$$\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{L^2}{4}.$$

The midpoint moves on a quarter-circle of radius $L/2$ centred at the origin, a fact I find mildly magical, where "magical" is what physicists call mathematics when they are not paying attention.



Question 1.16. Interviewer. If $(\sqrt[3]{4} - \sqrt[3]{2})(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}) = 2$, find abc.

SHELDON. Apply the identity $(x-y)(x^2+xy+y^2)=x^3-y^3$ with $x=\sqrt[3]{4}$, $y=\sqrt[3]{2}$:

$$x^3-y^3=4-2=2, \qquad x^2=\sqrt[3]{16}, \quad xy=\sqrt[3]{8}=2, \quad y^2=\sqrt[3]{4}.$$

Reading off, $(a,b,c)=(16,8,4)$ and

$$abc=16\cdot 8\cdot 4=512.$$

QUESTION 1.17. INTERVIEWER. As $n\to\infty$, which is larger: $2^{2^{2^n}}$ or 100^{100^n}?

SHELDON. Take $\log_2$ twice. First:

$$\log_2(2^{2^{2^n}})=2^{2^n}, \qquad \log_2(100^{100^n})=100^n\log_2 100.$$

Second:

$$\log_2(2^{2^n})=2^n, \quad \log_2(100^n\log_2 100)=n\log_2 100+\text{const}.$$

For large n, 2^n dominates any linear function of n, so the first tower is larger, and in fact grows incomparably faster. The tower wins by *height*, not by *base*.

QUESTION 1.18. INTERVIEWER. Show that if $(n-3)^3+n^3=(n+3)^3$ for some integer n, then n is even and $n^2\mid 54$. Deduce no such integer exists.

SHELDON. Expand:

$$(n+3)^3-(n-3)^3=n^3 \implies n^3=18n^2+54.$$

So $n^2(n-18)=54$, whence $n^2\mid 54$.

The perfect-square divisors of $54=2\cdot 3^3$ are $\{1,9\}$, so $n\in\{\pm1,\pm3\}$ (all odd). But $n^2(n-18)=54$ is even, and n^2 and $n-18$ share n's parity, so n must be even. The two conclusions are incompatible, so no integer satisfies the equation.

Question 1.19. Interviewer. Find $\sqrt{5-3\sqrt{2}}$.

Sheldon. Attempt a denesting $\sqrt{5-3\sqrt{2}} = \sqrt{a} - \sqrt{b}$ with $a > b > 0$. Squaring,

$$a + b - 2\sqrt{ab} = 5 - 3\sqrt{2} \implies a + b = 5, \quad 4ab = 18.$$

Thus a, b are roots of $t^2 - 5t + \frac{9}{2} = 0$, giving $t = \frac{5\pm\sqrt{7}}{2}$. Hence

$$\sqrt{5-3\sqrt{2}} = \sqrt{\tfrac{5+\sqrt{7}}{2}} - \sqrt{\tfrac{5-\sqrt{7}}{2}} \approx 0.870.$$

No denesting in $\mathbb{Q}[\sqrt{2}]$. Suppose $(a + b\sqrt{2})^2 = 5 - 3\sqrt{2}$ with $a, b \in \mathbb{Q}$. Expanding, $a^2 + 2b^2 = 5$ and $2ab = -3$. Then

$$(a^2 - 2b^2)^2 = (a^2 + 2b^2)^2 - 8a^2b^2 = 25 - 8 \cdot (9/4) = 7.$$

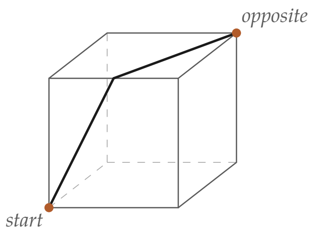
A rational square cannot equal 7, so no solution exists. The answer therefore cannot be written using only $\sqrt{2}$; the form above, inside $\mathbb{Q}[\sqrt{7}]$, is the simplest nested-radical representation.

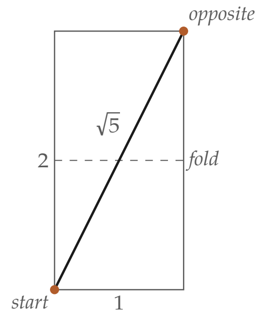
Question 1.20. Interviewer. An ant is at one vertex of a unit cube and may walk only on the faces. What is the shortest distance to the opposite vertex?

Sheldon. Unfold the cube. A surface path becomes a path in the unfolded planar development, and the shortest path between two points in a plane is a straight line. From one vertex to its antipode we cross at least two faces; each two-face unfolding is a 1×2 rectangle, whose diagonal has length

$$\sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.236.$$

Compare: walking along edges needs *three* of them to reach the antipode (each edge flips one coordinate, and the two vertices differ in all three), length 3; a face diagonal plus one edge gives $1 + \sqrt{2} \approx 2.414$. The unfolded geodesic $\sqrt{5}$ beats both. One of the few genuine "party tricks" mathematicians are permitted.





Question 1.21. Interviewer. Which is bigger, 2^x or x^2?

Sheldon. The question is ill-posed as stated, since the answer depends on x. I partition the real line and treat each region in turn.

$x > 0$. Take logarithms: $2^x > x^2 \iff x\ln 2 > 2\ln x \iff \frac{\ln 2}{2} > \frac{\ln x}{x}$. The function $g(x) = \frac{\ln x}{x}$ increases on $(0, e)$ and decreases on (e, ∞), crossing the horizontal level $\frac{\ln 2}{2}$ exactly at $x = 2$ and $x = 4$ (since $\frac{\ln 2}{2} = \frac{\ln 4}{4}$). Hence

$$2^x > x^2 \text{ on } (0, 2) \cup (4, \infty), \qquad 2^x < x^2 \text{ on } (2, 4),$$

with equality at $x = 2$ and $x = 4$.

$x = 0$. $2^0 = 1 > 0 = 0^2$.

$x < 0$. Here $2^x \in (0, 1)$ while $x^2 > 0$. The difference $h(x) = 2^x - x^2$ satisfies $h(0) = 1 > 0$ and $h(-1) = \frac{1}{2} - 1 < 0$, so by the intermediate value theorem there is at least one root $x_0 \in (-1, 0)$. Moreover $h'(x) = 2^x \ln 2 - 2x$: on $x < 0$, $2^x \ln 2 \in (0, \ln 2)$ and $-2x > 0$, so $h'(x) > 0$ throughout $(-\infty, 0)$, making h strictly increasing. The root is unique, and numerically $x_0 \approx -0.7667$. Therefore

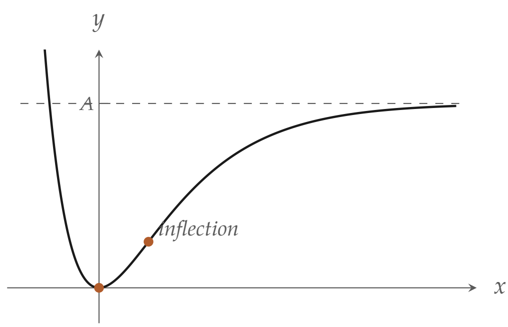
$$2^x < x^2 \text{ on } (-\infty, x_0), \qquad 2^x > x^2 \text{ on } (x_0, 0],$$

with equality only at $x = x_0$.

I trust my answer contains more intervals than the interviewer expected. Had they wanted a single inequality, they should have constrained the domain.

Question 1.22. Interviewer. Sketch $y = A(1 - e^{Bx})^2$.

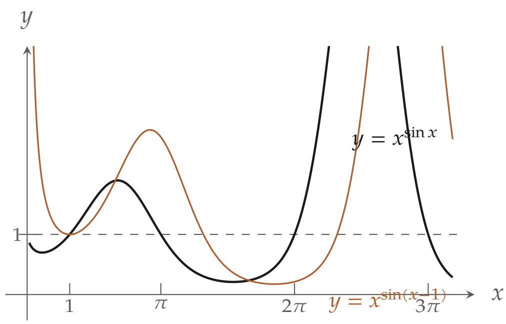
Sheldon. The two constants only set the orientation, so fix the representative case $A > 0$, $B < 0$ (the usual saturating transient) and read off the rest by symmetry: the sign of B reflects the graph in the y-axis (replace x by $-x$), and the sign of A scales it, reflecting in the x-axis when $A < 0$ (with $A = 0$ the degenerate flat line $y = 0$). Since $(1 - e^{Bx})^2 \geq 0$, the curve never crosses the axis for $A > 0$. In the representative case it passes through the origin ($y(0) = 0$), rises monotonically, and asymptotes $y \to A$ as $x \to \infty$. Differentiation: $y' = -2ABe^{Bx}(1 - e^{Bx}) > 0$ for $x > 0$, and $y' = 0$ only at $x = 0$. An inflection point occurs where $y'' = 0$, i.e. where $2e^{Bx} = 1$, giving $x = -\ln 2/B$. For $B < 0$, that is $x = \ln 2/|B| > 0$.



Question 1.23. Interviewer. Sketch $y = x^{\sin x}$ and $y = x^{\sin(x-1)}$.

Sheldon. Both are defined for $x > 0$. The family $y = x^{g(x)}$ rewrites as $y = e^{g(x)\ln x}$, which exceeds 1 when $g(x)\ln x > 0$, i.e. when $g(x)$ and $\ln x$ share a sign. Crossings of $y = 1$ occur where $g(x)\ln x = 0$: either $x = 1$, or $g(x) = 0$.

For $y = x^{\sin x}$, zeros of $\sin x$ are $x = k\pi$; crossings of $y = 1$ at $x = 1, \pi, 2\pi, 3\pi, \ldots$, oscillating above/below as $\sin x$ changes sign. For $y = x^{\sin(x-1)}$, shift by $+1$: crossings at $x = 1, 1+\pi, 1+2\pi, \ldots$. As $x \to 0^+$ the two part company, a fact the careless sketcher misses. For $x^{\sin x}$ the exponent $\sin x \approx x$ makes $\sin x \ln x \to 0$, so $y \to 1$. For $x^{\sin(x-1)}$ the exponent tends to $\sin(-1) \approx -0.84 < 0$, so $y = x^{(\text{negative})} \to +\infty$. Thus $x^{\sin x}$ creeps up to 1 while $x^{\sin(x-1)}$ rockets up the left edge; both then oscillate with growing amplitude.



Question 1.24. Interviewer. Find an approximation to $1^4 + 2^4 + \cdots + 100^4$.

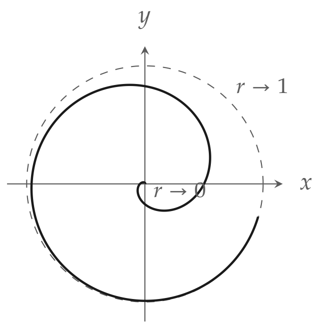
Sheldon. Approximate the sum by the integral $\int_0^{100} x^4\,dx = \frac{100^5}{5} = 2 \times 10^9$. For a tighter estimate, Faulhaber's formula gives the exact value

$$\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

which for $n = 100$ yields $\frac{100 \cdot 101 \cdot 201 \cdot 30299}{30} = 2\,050\,333\,330$. The integral 2×10^9 is within 2.5% of the exact answer; adequate for interview purposes and consistent with the general rule $\sum_{k=1}^{n} k^p \sim n^{p+1}/(p+1)$.

Question 1.25. Interviewer. Sketch the parametric curve $x(\theta) = \frac{e^\theta}{1+e^\theta}\cos\theta$, $y(\theta) = \frac{e^\theta}{1+e^\theta}\sin\theta$.

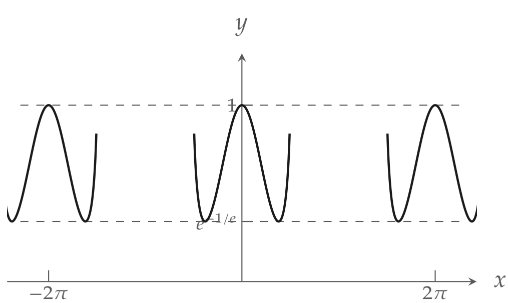
SHELDON. The radial distance from the origin is $r(\theta) = \frac{e^\theta}{1+e^\theta}$, a logistic (sigmoid) function: $r \to 0$ as $\theta \to -\infty$ and $r \to 1$ as $\theta \to +\infty$, with $r(0) = \frac{1}{2}$. The curve is a spiral in polar form $(r(\theta), \theta)$ whose radius saturates: it accumulates at the origin as $\theta \to -\infty$ and winds tightly onto the unit circle as $\theta \to +\infty$.



QUESTION 1.26. INTERVIEWER. Sketch $y = (\cos x)^{\cos x}$.

SHELDON. Real-valued only where $\cos x > 0$, i.e. on the intervals $(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi)$. Within each such interval, set $u = \cos x \in (0, 1]$: then $y = u^u$, the same x^x curve I sketched for Q12, attaining its minimum $y = e^{-1/e} \approx 0.692$ at $u = 1/e$ and maximum $y = 1$ at $u = 1$ (i.e. $x = 2k\pi$). As $x \to \frac{\pi}{2}^-$, $u \to 0^+$ and $y \to 1$.

The graph is therefore a 2π-periodic arrangement of "humps," each bounded between $y = e^{-1/e}$ and $y = 1$, with gaps where $\cos x \leq 0$.



Question 1.27. Interviewer. Prove that $\frac{a}{b} + \frac{b}{a} \geq 2$.

Sheldon. The inequality requires $ab > 0$ (otherwise it can fail or be undefined). Under that assumption, apply AM–GM to the two positive quantities $\frac{a}{b}$ and $\frac{b}{a}$:

$$\frac{a}{b} + \frac{b}{a} \geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{a}} = 2,$$

with equality iff $\frac{a}{b} = \frac{b}{a}$, i.e. $a^2 = b^2$. Since we are in the regime $ab > 0$, the root $a = -b$ (which needs $ab < 0$) is excluded, so equality holds iff $a = b$. Equivalently, $(a - b)^2 \geq 0$ gives $a^2 + b^2 \geq 2ab$; divide by $ab > 0$.

Had the interviewer not specified $ab > 0$, I would be obliged to point out that with $a = 1, b = -1$ the expression equals -2, a clear counterexample. Precision in hypotheses is not a luxury; it is the price of admission.

Question 1.28. Interviewer. Sketch $y = e^x$, $y = e^{-x}$, and $y = \frac{e^x + e^{-x}}{2} = \cosh x$ on the same axes.

Sheldon. e^x is the reflection of e^{-x} across the y-axis, and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ is their arithmetic mean, an even function with $\cosh 0 = 1$ as its minimum. All three curves pass through $(0, 1)$.

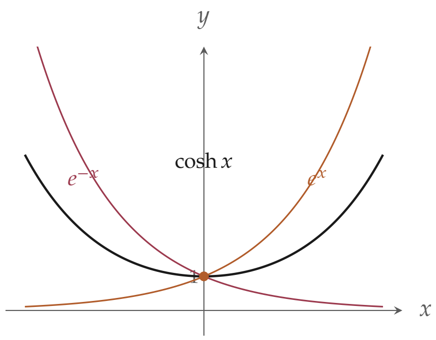
For $|x| > 0$ the identity

$$\cosh x = \frac{1}{2}(e^{|x|} + e^{-|x|})$$

gives, after subtraction, $\cosh x - \frac{1}{2}e^{|x|} = \frac{1}{2}e^{-|x|} > 0$ and $e^{|x|} - \cosh x = \frac{1}{2}(e^{|x|} - e^{-|x|}) > 0$ for $x \neq 0$. Hence the rigorous sandwich

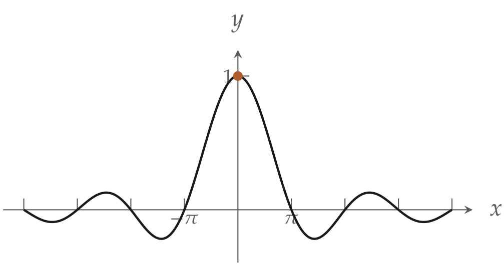
$$\frac{1}{2}e^{|x|} < \cosh x < e^{|x|} \qquad (x \neq 0),$$

with both inequalities tight in the ratio sense: $\cosh x/(\frac{1}{2}e^{|x|}) = 1 + e^{-2|x|} \to 1$ as $|x| \to \infty$, so $\cosh x$ is asymptotically half of $e^{|x|}$ on the dominant side.



Question 1.29. Interviewer. Sketch $y = \frac{\sin x}{x}$.

Sheldon. Even function: $\operatorname{sinc}(-x) = \operatorname{sinc}(x)$. Removable singularity at $x = 0$ with $\lim_{x\to 0} \operatorname{sinc}(x) = 1$ (first-order Taylor expansion of $\sin x$). Zeros at $x = k\pi$, $k \in \mathbb{Z} \setminus \{0\}$. The envelope is $\pm 1/|x|$, so oscillations decay in amplitude as $|x| \to \infty$. Local extrema occur where $\tan x = x$, not at the zeros of $\cos x$ as one might carelessly suppose.



Question 1.30. Interviewer. Prove that $n^7 - n$ is always divisible by 42.

Sheldon. $42 = 2 \cdot 3 \cdot 7$, so it suffices to show each of $2, 3, 7$ divides $n^7 - n$. Fermat's little theorem gives $n^p \equiv n \pmod p$ for every prime p, which handles the three primes as follows.

- $p = 7$: $n^7 \equiv n \pmod 7$ directly.
- $p = 3$: $n^3 \equiv n \pmod 3$, so $n^7 = (n^3)^2 \cdot n \equiv n^2 \cdot n = n^3 \equiv n \pmod 3$.

- $p = 2$: $n^2 \equiv n \pmod 2$, and repeated application gives $n^7 \equiv n \pmod 2$.

The three primes are pairwise coprime, so $42 \mid n^7 - n$. The cleaner statement is that for every prime p, $n^p - n$ is divisible by p; the Carmichael-like product $\prod_p p$ over primes p with $p - 1 \mid 7 - 1 = 6$, i.e. $p \in \{2, 3, 7\}$, gives the universal divisor 42. This is the beginning of a rather beautiful general pattern, but I shall refrain, as you did not ask.

QUESTION 1.31. INTERVIEWER. Explain why $(6 + \sqrt{37})^{20}$ is extremely close to an integer.

SHELDON. Consider the companion quantity $(6 - \sqrt{37})^{20}$. Since $\sqrt{37} \approx 6.0828$, we have $6 - \sqrt{37} \approx -0.0828$, whose twentieth power is vanishingly small (roughly 10^{-22} in absolute value, with sign $+$ since 20 is even).

The sum $(6 + \sqrt{37})^{20} + (6 - \sqrt{37})^{20}$ is an integer: expand by the binomial theorem and note that all odd-power-of-$\sqrt{37}$ terms cancel, leaving a rational number whose denominators are absorbed by the even powers of 37. Hence

$$(6 + \sqrt{37})^{20} = \underbrace{\left[(6 + \sqrt{37})^{20} + (6 - \sqrt{37})^{20}\right]}_{\text{integer}} - \underbrace{(6 - \sqrt{37})^{20}}_{\approx 10^{-22}},$$

which differs from the integer by about 10^{-22}, matching the ...9999999999437569 tail of the decimal expansion perfectly.

The same mechanism explains why $(1 + \sqrt{2})^n$ approaches integers for large n, and indeed the famous "almost integer" $e^{\pi\sqrt{163}}$ (Ramanujan's constant), though the latter requires class-field theory rather than the binomial theorem.

QUESTION 1.32. INTERVIEWER. Each point of the plane is coloured red, white, or blue. Prove that some two points at distance 1 share a colour.

SHELDON. Suppose for contradiction that no two unit-separated points share a colour. Take any equilateral triangle with side 1, vertices A, B, C: pairwise distances are all 1, so all three vertices must be differently coloured (one of each of red, white, blue).

Now reflect A across line BC to obtain A'. Then $BA' = CA' = 1$, so A' differs in colour from both B and C and hence must share A's colour. But $AA' = \sqrt{3}$. The same argument applied to every equilateral unit triangle with apex A shows that every point on the circle of radius $\sqrt{3}$ centred at A shares A's colour.

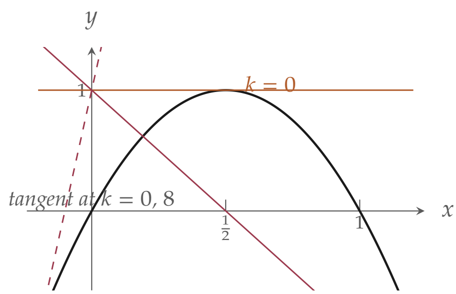
A circle of radius $\sqrt{3}$ contains chords of length 1 (any chord subtending angle $2 \arcsin \frac{1}{2\sqrt{3}}$). Such a chord gives two points, both coloured $c(A)$, at distance 1: contradiction. This problem descends to the celebrated Hadwiger–Nelson problem, which asks the chromatic number of the unit-distance graph of the plane. For 68 years it sat between 4 and 7, until Aubrey de Grey, the biologist with the beard, raised the lower bound to 5 in 2018.

QUESTION 1.33. INTERVIEWER. Draw $y = 4x(1-x)$ and $y = kx + 1$. For which values of k do the two curves cross twice?

SHELDON. The parabola $y = 4x(1-x)$ opens downward, peaks at $(\frac{1}{2}, 1)$, and crosses the x-axis at 0 and 1. Every line $y = kx+1$ passes through the point $(0, 1)$. Note this point is *not* on the parabola: at $x = 0$ the parabola is at $y = 0$, and indeed $(0, 1)$ lies strictly above the parabola, whose maximum value 1 is reached only at $x = \frac{1}{2}$. So $(0, 1)$ is an external point, and the line is free to meet the parabola twice, once, or not at all.

Setting $4x - 4x^2 = kx + 1$ gives $4x^2 + (k-4)x + 1 = 0$, whose discriminant is $(k-4)^2 - 16 = k^2 - 8k = k(k-8)$. Two distinct real crossings require $k(k-8) > 0$, i.e. $k < 0$ or $k > 8$.

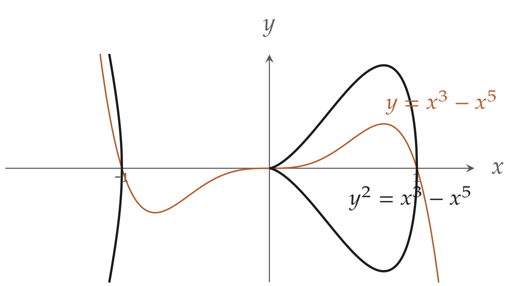
Geometrically: from the external point $(0, 1)$ there are exactly two tangents to the parabola, of slopes $k = 0$ and $k = 8$, touching at $(\frac{1}{2}, 1)$ and $(-\frac{1}{2}, -3)$ respectively. For k outside $[0, 8]$ the line is steeper than (or runs counter to) both tangents and cuts the parabola twice; for $0 < k < 8$ it passes between the tangents and misses the parabola entirely; at $k = 0$ and $k = 8$ it is tangent, a single contact.



Question 1.34. Interviewer. Sketch $y = x^3 - x^5$, hence sketch $y^2 = x^3 - x^5$.

Sheldon. For $y = x^3 - x^5 = x^3(1 - x^2)$: odd function; zeros at $x \in \{-1, 0, 1\}$; local max/min where $y' = 3x^2 - 5x^4 = 0$, i.e. at $x = 0$ (inflection) and $x = \pm\sqrt{3/5}$, giving local max $y(\sqrt{3/5}) = (3/5)^{3/2}(1 - 3/5) = \frac{2}{5}\sqrt{\frac{27}{125}}$ and its negative.

For $y^2 = x^3(1 - x^2)$: requires $x^3(1 - x^2) \geq 0$, i.e. x^3 and $1 - x^2$ agree in sign, which happens on $[0, 1]$ and on $(-\infty, -1]$. On $[0, 1]$ the curve is a closed loop (both signs of y), and on $(-\infty, -1]$ it is an unbounded pair of branches extending rightward-to-left.



Question 1.35. Interviewer. Which is larger, $8!^{1/8}$ or $9!^{1/9}$?

Sheldon. Compare by taking logarithms: equivalently, compare $\frac{\ln 8!}{8}$ and $\frac{\ln 9!}{9}$, i.e. $\frac{\ln 9!}{\ln 8!} \overset{?}{\gtrless} \frac{9}{8}$, i.e.

$$\frac{\ln 8! + \ln 9}{\ln 8!} \overset{?}{\gtrless} \frac{9}{8} \iff \frac{\ln 9}{\ln 8!} \overset{?}{\gtrless} \frac{1}{8}.$$

Now $\ln 8! = \ln(8!) \approx \ln(40320) \approx 10.605$, and $\ln 9 \approx 2.197$, so $\frac{\ln 9}{\ln 8!} \approx 0.2072 > \frac{1}{8} = 0.125$. Hence $9!^{1/9} > 8!^{1/8}$.

More generally, $(n!)^{1/n}$ is strictly increasing in n; this follows from AM–GM applied to the positive multiset $\{1, 2, \ldots, n+1\}$, or from Stirling's approximation $(n!)^{1/n} \sim n/e$.

Question 1.36. Interviewer. Is $x^2 + 2$ ever divisible by 5?

Sheldon. The squares modulo 5 are $\{0, 1, 4\}$ (since $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 4$, $4^2 = 1$). Hence $x^2 + 2 \pmod 5 \in \{2, 3, 1\}$, never 0. So $5 \nmid x^2 + 2$ for any integer x.

Equivalently, $-2 \equiv 3 \pmod 5$ is not a quadratic residue: 3 is absent from $\{0, 1, 4\}$. By the law of quadratic reciprocity, -2 is a QR modulo p iff $p \equiv 1, 3 \pmod 8$; for $p = 5$, $5 \equiv 5 \pmod 8$, so -2 is a non-residue. You did not require the general statement. I produce it for rigour.

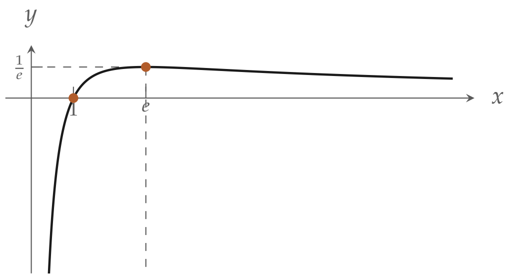
Question 1.37. Interviewer. Sketch $y = \dfrac{\ln x}{x}$ and find its maximum.

Sheldon. Differentiate: $y' = \frac{1 - \ln x}{x^2}$, positive on $(0, e)$ and negative on (e, ∞), vanishing at $x = e$. Hence y strictly increases on $(0, e)$ and strictly decreases on (e, ∞), so the critical point at $x = e$ is the unique global maximum on $(0, \infty)$:

$$y(e) = \frac{1}{e}.$$

Endpoint behaviour: $y \to -\infty$ as $x \to 0^+$ (logarithm dominates the vanishing denominator x), and $y \to 0^+$ as $x \to \infty$ (denominator dominates). There is a single zero at $x = 1$. The monotone-then-monotone structure combined with the two limits at the endpoints rules out any further local extrema.

This is the same $f(x) = \ln x / x$ that settled the e^π vs. π^e question earlier; its one-humped shape is the entire content of that comparison.



Question 1.38. Interviewer. A bag has 5 blue balls and 1 red ball. You draw balls one at a time and discard. What is the probability that the last ball drawn is red?

Sheldon. By symmetry, the red ball is equally likely to end up in any of the 6 positions in the draw sequence. The probability it is the last one is therefore $\frac{1}{6}$.

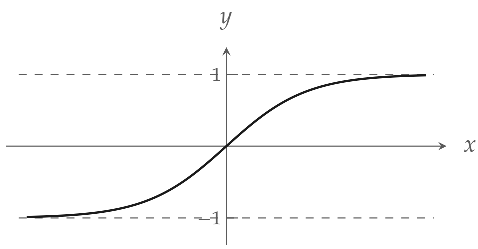
A more pedestrian calculation: the probability that the first draw is not red is $\frac{5}{6}$; given that, the second draw is not red with probability $\frac{4}{5}$; and so on. Multiplying,

$$\frac{5}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

which is the same answer, obtained with considerably more effort. The symmetry argument is what a mathematician would call "elegant" and I would call "the only calculation worth performing."

Question 1.39. Interviewer. Sketch $y = \dfrac{e^x - 1}{e^x + 1}$.

Sheldon. The function simplifies to $y = \tanh(x/2)$: divide numerator and denominator by $e^{x/2}$ to obtain $\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}$. Odd function, $y(0) = 0$; horizontal asymptotes $y \to 1$ as $x \to \infty$ and $y \to -1$ as $x \to -\infty$; strictly increasing everywhere; inflection at the origin.



Question 1.40. Interviewer. Integrate $y = \dfrac{1}{x^2}$.

Sheldon.

$$\int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C.$$

For definite integrals over any interval not containing 0, this is unproblematic. Over an interval containing 0 the integrand is not Lebesgue-integrable (the improper integral $\int_{-1}^{1} x^{-2}\,dx$ diverges), and the Cauchy principal value does not help here because the function is everywhere positive. Anyone claiming the integral equals $-\frac{1}{x}|_{-1}^{1} = -2$ has committed what I classify as a *Category III Error*: syntactically valid, semantically wrong.

Question 1.41. Interviewer. There are n teams in a tournament. Each match eliminates one team. How many matches before one team remains?

Sheldon. Bijection: each match eliminates exactly one team, and we must eliminate $n - 1$ teams to leave a single champion. Hence $n-1$ matches. Conservation of teams, phrased as combinatorics. The format of the bracket is immaterial: single-elimination, double-elimination, Swiss, or a Klingon *kal'Hyah* tournament all require $n - 1$ eliminations.

Question 1.42. Interviewer. By examining $f(x) = \dfrac{\ln x}{x}$, decide which is greater, e^x or x^e.

Sheldon. For $x > 0$ the inequality $e^x > x^e$ is equivalent (after taking logarithms) to $x > e \ln x$, that is to $\frac{\ln x}{x} < \frac{1}{e}$. Differentiating gives $f'(x) = (1 - \ln x)/x^2$, which is positive on $(0, e)$ and negative on (e, ∞), so f has a unique global maximum at $x = e$ with $f(e) = 1/e$. For every $x > 0$ with $x \neq e$ the inequality $f(x) < 1/e$ is therefore strict, and

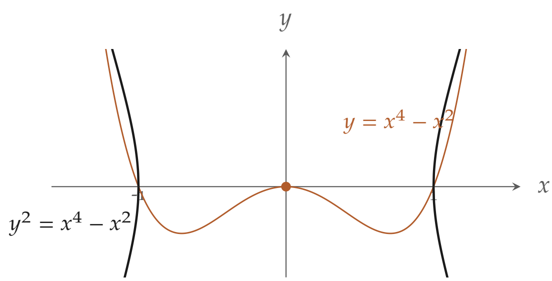
$$e^x > x^e \quad \text{for all } x > 0,\ x \neq e, \qquad e^x = x^e \text{ at } x = e.$$

Verification at $x = 3$: $e^3 \approx 20.086$ and $3^e \approx 19.812$, in agreement. The e^π versus π^e question (Q10) is the special case $x = \pi$.

Question 1.43. Interviewer. Sketch $y = x^4 - x^2$. What does it look like zoomed at the origin? Then sketch $y^2 = x^4 - x^2$.

Sheldon. For $y = x^4 - x^2 = x^2(x^2 - 1)$: even, zeros at 0 (double) and ± 1, minima at $x = \pm\frac{1}{\sqrt{2}}$ of value $-\frac{1}{4}$, and $y \to +\infty$ as $|x| \to \infty$. Near the origin, Taylor expansion gives $y \approx -x^2$, so the graph locally resembles a downward parabola $y = -x^2$ with the quartic correction invisible.

For $y^2 = x^4 - x^2 = x^2(x^2 - 1)$: requires $x^2(x^2 - 1) \geq 0$, i.e. $|x| \geq 1$ (together with the isolated point $x = 0$). The curve consists of two unbounded arms for $x \geq 1$ and $x \leq -1$ (each symmetric about the x-axis), plus the single point at the origin.



Question 1.44. Interviewer. Solve $x^4+3x^3-2x^2-3x+1=0$.

Sheldon. The polynomial is *palindromic* (coefficients $1, 3, -2, 3, 1$ read the same forwards and backwards up to sign of alternate terms, actually $1, 3, -2, -3, 1$, which is *anti-palindromic* in the odd-degree coefficients). Divide by x^2:

$$x^2 + 3x - 2 - \frac{3}{x} + \frac{1}{x^2} = 0 \iff \left(x^2 + \frac{1}{x^2}\right) + 3\left(x - \frac{1}{x}\right) - 2 = 0.$$

Let $y = x - \frac{1}{x}$; then $x^2 + \frac{1}{x^2} = y^2 + 2$. The equation becomes $y^2 + 3y = 0 = y(y+3)$, so $y = 0$ or $y = -3$.

Case $y = 0$: $x - \frac{1}{x} = 0 \implies x^2 = 1 \implies x = \pm 1$.

Case $y = -3$: $x - \frac{1}{x} = -3 \implies x^2 + 3x - 1 = 0 \implies x = \frac{-3\pm\sqrt{13}}{2}$.

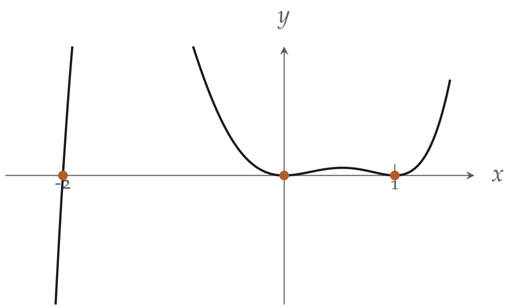
The four roots are ± 1 and $\frac{-3\pm\sqrt{13}}{2}$.

Question 1.45. Interviewer. What is the last digit of 3^{50}?

Sheldon. The last digits of 3^n for $n = 1, 2, 3, 4$ are $3, 9, 7, 1$, then the cycle of length 4 repeats. Since $50 \equiv 2 \pmod 4$, the last digit of 3^{50} is that of 3^2, namely 9.

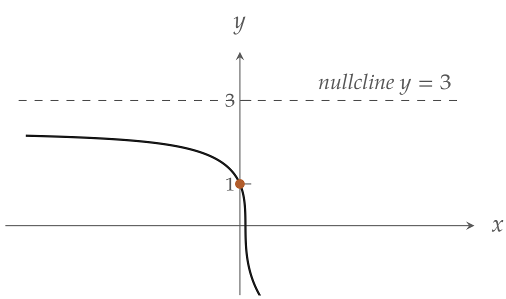
Question 1.46. Interviewer. Sketch $y = x^5 - 3x^3 + 2x^2$.

Sheldon. Factor: $y = x^2(x^3 - 3x + 2) = x^2(x-1)(x^2+x-2) = x^2(x-1)^2(x+2)$. Zeros: $x = 0$ (double), $x = 1$ (double), $x = -2$ (simple). A double zero produces tangency with the x-axis; a simple zero produces a crossing. As $x \to -\infty$, $y \to -\infty$; as $x \to +\infty$, $y \to +\infty$. Between -2 and 0 the function is positive (crossing up at -2, tangent down to zero at 0); between 0 and 1 positive (tangent from zero at 0, tangent to zero at 1); above 1, positive and rising.



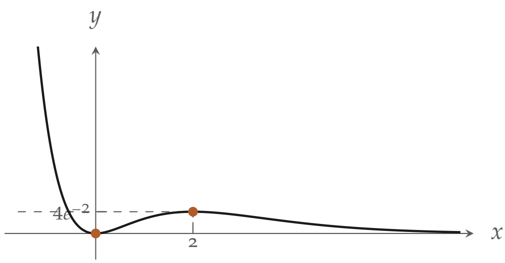
Question 1.47. Interviewer. Sketch the solution curve of $\dfrac{dy}{dx} = \dfrac{y-3}{x^2+y^2}$ through $(0, 1)$.

Sheldon. Qualitative analysis first. The slope vanishes on the horizontal line $y = 3$ (nullcline) and is undefined only at the origin, where $x^2 + y^2 = 0$. For $y < 3$, the numerator is negative and the denominator positive, so the slope is negative: y decreases as x increases. Hence the solution through $(0, 1)$ descends to the right and rises to the left. As $|x| \to \infty$ with y bounded, the slope $\sim (y-3)/x^2 \to 0$, so the curve flattens out at both ends into horizontal asymptotes.



Question 1.48. Interviewer. Sketch $y = x^2 e^{-x}$.

Sheldon. Differentiate: $y' = (2x - x^2)e^{-x} = x(2-x)e^{-x}$, vanishing at $x = 0$ (local minimum, also a zero of y) and $x = 2$ (local maximum, value $4e^{-2} \approx 0.541$). As $x \to +\infty$, $y \to 0^+$ exponentially; as $x \to -\infty$, both x^2 and e^{-x} blow up, so $y \to +\infty$.

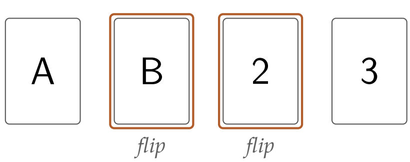


Question 1.49. Interviewer. Cards show a letter on one side and a number on the other. The rule is: if the number is even, the letter must be a vowel. Four cards on the table show A, B, 2, 3. Which must be flipped to verify the rule?

Sheldon. The Wason selection task. The rule is the conditional even ⇒ vowel, with contrapositive consonant ⇒ odd. A card falsifies the rule only if it exhibits an even number paired with a consonant. Flip exactly the cards capable of producing that witness:

- A (vowel): irrelevant; the rule does not constrain what number a vowel hides.
- B (consonant): flip; the contrapositive requires an odd number.
- 2 (even): flip; the rule requires a vowel.
- 3 (odd): irrelevant; the rule is silent on odd numbers.

Answer: B and 2.



Approximately 90% of adult humans answer A and 2, which is why I prefer the company of Leonard; he gets this right about 47% of the time, which is still 47 percentage points above a fair coin and meaningfully below my own performance.

Question 1.50. Interviewer. Integrate $\cos^2 x$ and $\cos^3 x$.

Sheldon. For $\cos^2 x$, use the double-angle identity $\cos^2 x = \frac{1+\cos 2x}{2}$:

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

For $\cos^3 x$, pull out one factor and substitute $u = \sin x$, $du = \cos x \, dx$:

$$\int \cos^3 x \, dx = \int (1 - \sin^2 x) \cos x \, dx = \sin x - \frac{\sin^3 x}{3} + C.$$

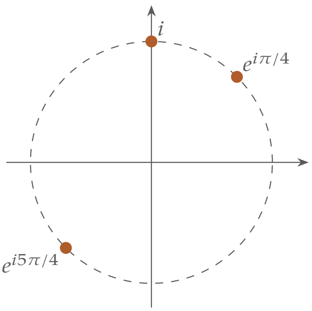
The general pattern: even powers reduce by double-angle, odd powers reduce by $u = \sin x$. This bifurcation is why a physicist and a mathematician will never agree on the most natural way to compute $\int \cos^n x \, dx$; it depends on the parity of n, and physicists dislike parity-dependent derivations on principle.

Question 1.51. Interviewer. What is the square root of i?

Sheldon. Write $i = e^{i\pi/2}$; square roots come from halving the argument (and shifting by π for the other root):

$$\sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}} \quad \text{or} \quad e^{i5\pi/4} = -\frac{1+i}{\sqrt{2}}.$$

Verification: $\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{(1+i)^2}{2} = \frac{2i}{2} = i.$



Question 1.52. Interviewer. Integrate $\ln x$.

Sheldon. Integration by parts with $u = \ln x$, $dv = dx$ (so $du = dx/x$, $v = x$):

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x + C.$$

The trick, if one insists on calling it a trick, is to recognise that $\ln x$ is a perfectly good u even without an explicit dv.

Question 1.53. Interviewer. A cube is painted with 6 different colours, one per face. How many distinguishable colourings are there? What if there are n colours?

Sheldon. The cube's rotation group has 24 elements (six face-up choices, each with four rotations about the vertical axis). With all 6 colours distinct and each used exactly once, the number of face-labellings is 6!, of which each equivalence class under rotation contains exactly 24:

$$\text{Distinguishable colourings} = \frac{6!}{24} = \frac{720}{24} = 30.$$

For n distinct colours assigned to the 6 faces (each face a different colour, drawn from the palette of n): choose the 6-subset and assign, giving $\binom{n}{6} \cdot 6!$ ordered labellings, and divide by the rotation group:

$$\frac{\binom{n}{6} \cdot 6!}{24} = \frac{n!}{24 \cdot (n-6)!} = 30\binom{n}{6}.$$

If colours may repeat, the problem becomes a proper Burnside calculation summing fixed-point counts over the 24 rotations; I shall not expand it unless you ask, and you have not.

Question 1.54. Interviewer. Prove that $\sqrt{2}$ is irrational.

Sheldon. Suppose for contradiction that $\sqrt{2} = \frac{p}{q}$ with $\gcd(p, q) = 1$. Squaring, $2q^2 = p^2$, so $2 \mid p^2$ and hence $2 \mid p$ (since 2 is prime). Write $p = 2m$: then $2q^2 = 4m^2$, i.e. $q^2 = 2m^2$, so $2 \mid q^2$ and $2 \mid q$. But then $2 \mid \gcd(p, q) = 1$, absurd. The same argument generalises to $\sqrt{k}$ for any non-square positive integer k, via unique factorisation and a parity-of-exponents argument. I offer this in case the interviewer intended a follow-up; they need not.

Question 1.55. Interviewer. Three girls and four boys stand in a circle. What is the probability that exactly two girls are adjacent to each other, and the third girl is not adjacent to that pair?

Sheldon. Total circular arrangements: $6! = 720$ (fix one person's rotation). Direct counting is cleaner than inclusion–exclusion here.

Arrangements with no two girls adjacent. Seat the four boys first: $3! = 6$ circular arrangements. They create four gaps between consecutive boys, and the three girls must occupy three distinct gaps (at most one girl per gap, to avoid girl-girl adjacency). Choose 3 of 4 gaps and place the girls: $\binom{4}{3} \cdot 3! = 24$. Total: $6 \cdot 24 = 144$.

Arrangements with all three girls consecutive. Treat the three girls as a block of three (internally ordered $3! = 6$ ways). The block plus four boys is five circular units: $(5-1)! = 24$ arrangements. Total: $6 \cdot 24 = 144$.

Arrangements with at least one girl-pair adjacent: $720 - 144 = 576$. Of these, 144 have all three consecutive, so exactly one adjacent pair (with the third girl not adjacent to the block) accounts for

$$576 - 144 = 432, \qquad P = \frac{432}{720} = \tfrac{3}{5}.$$

Question 1.56. Interviewer. Why is a number of the form $\overline{abcabc}$ always divisible by 13?

Sheldon. Write $\overline{abcabc} = \overline{abc} \cdot 1000 + \overline{abc} = 1001 \cdot \overline{abc}$. Since $1001 = 7 \cdot 11 \cdot 13$, the number is divisible by 7, 11, *and* 13. The interviewer asked only for 13; I have provided three for the price of one.

Question 1.57. Interviewer. If I win a point with probability p, what is my probability of winning a game of tennis?

Sheldon. Let $q = 1 - p$. A game is won by the first player to reach four points with a lead of two; if tied at 3–3 ("deuce"), play continues until one player leads by two.

Non-deuce wins. Winning 4–k for $k \in \{0, 1, 2\}$ requires the player to win the final point and accumulate 3 wins against k losses among the preceding $3 + k$ points:

$$P(\text{win without deuce}) = \sum_{k=0}^{2} \binom{3+k}{k} p^4 q^k = p^4(1 + 4q + 10q^2).$$

Deuce branch. The probability of reaching 3–3 is $\binom{6}{3} p^3 q^3 = 20p^3q^3$. From deuce the probability of eventually winning is $r = \frac{p^2}{p^2+q^2}$ (solve the one-step recurrence $r = p^2 + 2pq \cdot r$).

Combining,

$$P(\text{win game}) = p^4(1 + 4q + 10q^2) + 20p^3q^3 \cdot \frac{p^2}{p^2 + q^2}.$$

At $p = \frac{1}{2}$ this collapses to $\frac{1}{2}$ by symmetry, a sanity check I recommend one always perform before presenting a closed form.

Question 1.58. Interviewer. Suggest prime factors of 612 612 503 503.

Sheldon. Observe the repeating-block structure: the number is $\overline{612\,612} \cdot 10^6 + \overline{503\,503}$, and each six-digit block is itself of the form $\overline{abcabc} = \overline{abc} \cdot 1001$. So

$$612\,612\,503\,503 = 1001 \cdot (612 \cdot 10^6 + 503) = 1001 \cdot 612\,000\,503.$$

Since $1001 = 7 \cdot 11 \cdot 13$, the primes 7, 11, and 13 are all factors of N. The remaining co-factor 612 000 503 yields no small prime up to 100 by direct trial division, so the interview-ready answer stops here, with the witty suggestion that 1001 alone supplies three primes.

Question 1.59. Interviewer. How many faces are there on an icosahedron?

Sheldon. 20. The regular icosahedron has $V = 12$ vertices, $E = 30$ edges, $F = 20$ faces (all equilateral triangles), satisfying Euler's formula $V - E + F = 12 - 30 + 20 = 2$, as any convex polyhedron must. The name itself, from Greek *eikosi* meaning twenty and *hedra* meaning seat, encodes the answer. I mention this because the interviewer's question could have been asked in Greek and still answered by translation alone.

Question 1.60. Interviewer. Prove that the product of four consecutive integers is divisible by 24.

Sheldon. Two proofs, the second of which is by elegance:

Direct. Among any four consecutive integers, exactly two are even, and those two differ by 2, so one is $\equiv 2 \pmod 4$ and the other is $\equiv 0 \pmod 4$; their product is divisible by 8. Among any three consecutive integers exactly one is divisible by 3, so among four consecutive there is at least one multiple of 3. Hence the product is divisible by $8 \cdot 3 = 24$. *Slick.* For any positive integer n,

$$n(n+1)(n+2)(n+3) = 4! \cdot \binom{n+3}{4},$$

which makes $24 \mid n(n+1)(n+2)(n+3)$ a restatement of the fact that binomial coefficients are integers. The same argument shows $k!$ divides the product of any k consecutive integers, a result one should know without proof by the time one is admitted.

Question 1.61. Interviewer. Using the digits 1 through 6 exactly once each, how many distinct six-digit numbers can you form? How many of these contain 1, 2, 3 in that order and adjacent? In how many does 5 occur before 1? How many six-digit numbers use all of the digits 1 through 5 (with repetition)?

SHELDON. Four parts, each a one-liner.

Distinct arrangements. $6! = 720$.

$1, 2, 3$ *adjacent and in that order.* Treat "123" as a single block. The block plus the digits $4, 5, 6$ is four items, permuted $4! = 24$ ways.

5 *before* 1. By symmetry, 5 precedes 1 in exactly half of all arrangements: $6!/2 = 360$.

Six-digit strings using all of $\{1, 2, 3, 4, 5\}$. By inclusion–exclusion on the forbidden events E_i ("digit i absent"),

$$\begin{aligned}\#\{\text{all five used}\} &= \sum_{k=0}^{5}(-1)^k\binom{5}{k}(5-k)^6 \\ &= 15625 - 20480 + 7290 - 640 + 5 = 1800.\end{aligned}$$

QUESTION 1.62. INTERVIEWER. The power set of an n-element set has what cardinality? How many of its subsets contain a given element x_1?

SHELDON. Each subset is determined by independently including or excluding each of the n elements: $|\mathcal{P}(S)| = 2^n$. Subsets containing x_1 are in bijection with subsets of the remaining $n-1$ elements (take the subset and adjoin x_1), so there are 2^{n-1} of them, i.e. exactly half.

This is incidentally the cleanest proof that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$: double-count the power set by size of subset.

QUESTION 1.63. INTERVIEWER. n people sit at a round table. What is the probability that person A sits exactly k seats from person B?

SHELDON. Fix A's seat. The other $n-1$ seats are equally likely destinations for B. "Exactly k seats away" (measured along the circle) has two positions (k clockwise and k anticlockwise) for

$1 \le k < n/2$, and one position when $k = n/2$ (only when n is even). Therefore

$$P = \begin{cases} \dfrac{2}{n-1} & 1 \le k < \frac{n}{2}, \\ \dfrac{1}{n-1} & k = \frac{n}{2},\ n \text{ even}. \end{cases}$$

Question 1.64. Interviewer. How can you maximise the number of regions n straight lines divide the plane into? What is the maximum? What if we replace lines with circles? What does this tell us about Venn diagrams?

Sheldon. *Lines.* Place the lines in general position: no two parallel, no three concurrent. When the kth line is added, it meets each of the previous $k-1$ lines in a distinct point, so it is chopped into k pieces, each of which splits one old region into two. Starting from 1 region (the empty plane), after n lines we have

$$L(n) = 1 + \sum_{k=1}^{n} k = 1 + \binom{n+1}{2} = \binom{n}{2} + n + 1.$$

Circles. Analogously, but the first circle is special: with no others to meet, it simply splits the plane into 2 regions. Each later circle, the kth for $k \ge 2$, meets the previous $k-1$ circles in up to $2(k-1)$ points, so it is cut into $2(k-1)$ arcs, each splitting one region. Hence

$$C(n) = 2 + \sum_{k=2}^{n} 2(k-1) = 2 + 2 \cdot \frac{(n-1)n}{2} = n^2 - n + 2.$$

Venn. An n-set Venn diagram requires 2^n regions. Circles give at most $n^2 - n + 2$: for $n = 3$, both equal 8, which is why three-circle Venns work. For $n = 4$, $2^4 = 16$ but $n^2 - n + 2 = 14$; so four-circle Venns cannot exist, and one must resort to ellipses or worse.

Question 1.65. Interviewer. How many vertices and edges does a line segment, square, cube, tesseract have? Conjecture the formulas for an n-dimensional hypercube. Give the coordinates of the tesseract's vertices. What is the longest diagonal?

SHELDON. The n-cube $[0,1]^n$ has vertices at $\{0,1\}^n$, so $|V| = 2^n$. An edge corresponds to a pair of vertices differing in exactly one coordinate: pick the coordinate (n choices), pick the fixed values on the remaining coordinates (2^{n-1} ways), giving

$$|E| = n \cdot 2^{n-1}.$$

For $n = 1,2,3,4$: $(V,E) = (2,1),(4,4),(8,12),(16,32)$. Tesseract vertices: $\{0,1\}^4$; longest diagonal is from $(0,0,0,0)$ to $(1,1,1,1)$, of length $\sqrt{4} = 2$. More generally, the main diagonal of an n-cube has length $\sqrt{n}$.

Bonus (since you asked for formulas): the number of k-dimensional faces of the n-cube is $\binom{n}{k}2^{n-k}$, by choosing which k coordinates vary and which $n-k$ are fixed to specific values. Summing gives $(1+2)^n = 3^n$: the "face polynomial," a minor delight.

QUESTION 1.66. INTERVIEWER. In how many ways can $2n$ players be paired into n first-round matches? Give a succinct expression for large n.

SHELDON. Line all $2n$ players up in some order ($(2n)!$ ways) and then pair position 1 with position 2, position 3 with position 4, and so on. This overcounts each pairing: within each of the n pairs the two players can be swapped (2^n orderings) and the n pairs themselves can be permuted ($n!$ orderings). Hence

$$\frac{(2n)!}{2^n \cdot n!} = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1),$$

the *double factorial* of $2n-1$. A succinct expression and a rare occasion on which a mathematician has bothered to define a useful shorthand.

Question 1.67. Interviewer. How many distinct regular-polygon tessellations of the plane exist? Why are there only five Platonic solids? Show that a Platonic solid with triangular faces must have 4, 8, or 20 faces.

Sheldon. *Tessellations.* A regular n-gon has interior angle $\frac{(n-2)\pi}{n}$. For k polygons to meet around a vertex without gap or overlap, $k \cdot \frac{(n-2)\pi}{n} = 2\pi$, i.e. $(n-2)k = 2n$, i.e. $k = 2 + \frac{4}{n-2}$. Integer solutions: $n \in \{3, 4, 6\}$ (giving $k = 6, 4, 3$). Three tessellations: triangles, squares, hexagons.

Platonic solids. At each vertex of a Platonic solid, at least three regular polygons meet, and their angles must sum to strictly less than 2π (otherwise the vertex is flat or intersects itself). For faces with n sides and k meeting at each vertex, $k \cdot \frac{(n-2)\pi}{n} < 2\pi$, i.e. $(n-2)(k-2) < 4$. Integer solutions with $n \geq 3$, $k \geq 3$: $(n, k) \in \{(3,3), (3,4), (3,5), (4,3), (5,3)\}$, yielding the tetrahedron, octahedron, icosahedron, cube, and dodecahedron respectively.

Triangular-faced solids. With $n = 3$ (triangles) and k meeting per vertex ($k \in \{3, 4, 5\}$), Euler's $V - E + F = 2$ and the incidences $V \cdot k = 3F$, $2E = 3F$ combine to

$$F = \frac{4k}{6-k} \in \{4, 8, 20\}.$$

Question 1.68. Interviewer. The numbers 1 to 1000 are written on a board. You pick two, a and b, and replace them with $|a-b|$. You continue until one number remains. Can that number be 1?

Sheldon. Consider the parity of the sum. Replacing (a, b) by $|a-b|$ changes the total by $-(a+b) + |a-b|$, which equals either $-2\min(a, b)$ (even) or $-2b$ (even); in either case the total's parity is preserved.

Initial sum: $1 + 2 + \cdots + 1000 = \frac{1000 \cdot 1001}{2} = 500\,500$, which is even. The final single number is therefore even; 1 is odd; impossible.

This is a delightful instance of the physicist's *conservation law* disguised as combinatorics. I was not asked which even numbers *are* achievable, and I shall resist the temptation to overclaim: since $|a - b| \leq \max(a, b)$, no operation ever produces a number larger than the largest already present, so the final value can never exceed 1000. The reachable values are precisely the even integers in $[0, 1000]$, not, as one is tempted to blurt, anything up to the initial sum $500\,500$.

Question 1.69. Interviewer. If you paint three faces of a cube red and the other three blue, how many distinguishable colourings are there?

Sheldon. By Burnside's lemma, the count of distinguishable colourings equals the average number of fixed colourings over the cube's rotation group (order 24).

- *Identity* (1): fixes all $\binom{6}{3} = 20$ colourings.
- *Face-axis* 180° (3 rotations): two fixed faces, two pairs of swapped side faces. For a fixed colouring each pair is monochromatic. Requiring three reds means (top+bottom red count) must be odd $= 1$, giving 2 choices for which of top/bottom is red, times 2 for which side-pair is red: 4 each, total 12.
- *Face-axis* $\pm 90°$ (6): side faces form a 4-cycle; fixed colouring requires all four sides same colour, which violates the 3–3 split. 0.
- *Edge-axis* 180° (6): three 2-cycles of faces, all swapped pairs. Each pair monochromatic; 3 pairs contribute $0, 2, 4$, or 6 reds; none equals 3. 0.
- *Vertex-axis* $\pm 120°$ (8): two 3-cycles of faces. Each cycle monochromatic. Three reds means one cycle red, one blue: 2 assignments per rotation, total 16.

Summing and averaging, $\frac{20+12+0+0+16}{24} = \frac{48}{24} = 2$. The two types are: three red faces sharing a vertex, and three red faces forming a "belt" (two opposite faces plus one adjacent face).

Question 1.70. Interviewer. Every subset of $\{1, 2, \dots, n\}$ either contains 1 or does not. Use this to prove $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$. Then explain why $\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}$.

Sheldon. *Pascal's identity.* An r-subset of $\{1, 2, \dots, n\}$ either contains 1 or does not. If it contains 1, complete it with $r-1$ further elements from $\{2, \dots, n\}$: $\binom{n-1}{r-1}$ ways. If not, pick all r from $\{2, \dots, n\}$: $\binom{n-1}{r}$ ways. Summing gives $\binom{n}{r}$.

Generalisation. Apply Pascal twice:

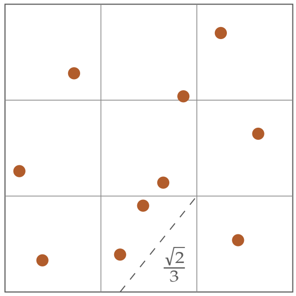
$$\begin{aligned}\binom{n}{r} &= \binom{n-1}{r-1} + \binom{n-1}{r} \\ &= [\binom{n-2}{r-2} + \binom{n-2}{r-1}] + [\binom{n-2}{r-1} + \binom{n-2}{r}] \\ &= \binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r}.\end{aligned}$$

Combinatorially: classify each r-subset by how many of $\{1, 2\}$ it contains (namely 0, 1, or 2), giving the three terms $\binom{n-2}{r}$, $2\binom{n-2}{r-1}$, $\binom{n-2}{r-2}$.

Question 1.71. Interviewer. Ten distinct points lie within a unit square. Prove that two of them lie within $\sqrt{2}/3$ of each other.

Sheldon. Partition the unit square into a 3×3 grid of nine smaller squares, each of side $\frac{1}{3}$. By pigeonhole, two of the ten points share a small square. The greatest distance between two points in a square of side $\frac{1}{3}$ is the diagonal, of length

$$\sqrt{\left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2} = \frac{\sqrt{2}}{3}.$$



Question 1.72. Interviewer. An infinite chessboard is filled with positive integers, each equal to the average of its four orthogonal neighbours. Show that all the integers are equal.

Sheldon. Since the entries are positive integers, the set of values attained is a non-empty subset of $\mathbb{N}$ bounded below, hence has a minimum m, attained at some cell C. The neighbours $N_1, \ldots, N_4$ of C satisfy $N_i \geq m$ and $\frac{N_1+N_2+N_3+N_4}{4} = m$. The only way four quantities each at least m average to m is for each to equal m. So m is also attained at all four neighbours of C.

Iterating: the set of cells with value m is closed under passage to neighbours, hence equals the entire (connected) chessboard. All entries equal m. This is the discrete maximum principle for harmonic functions, demonstrated a century before harmonic functions were named. I should add that the hypothesis of *positive integers* is necessary for the existence of a minimum; for real-valued harmonic functions on an infinite lattice one needs Liouville-style growth conditions.

Question 1.73. Interviewer. Two decks are shuffled independently and placed side by side. The top cards are paired, then the next, and so on. What is the probability of exactly k matching pairs, for $k \in \{0, 1, \ldots, 52\}$?

SHELDON. The joint shuffle is equivalent to one deck fixed and the other uniformly permuted. Matches correspond to fixed points of a uniform random permutation of 52 elements.

Number of permutations of n items with exactly k fixed points: choose the k fixed positions ($\binom{n}{k}$), then derange the remaining $n-k$ (D_{n-k} ways):

$$P(\text{exactly } k \text{ matches}) = \frac{\binom{n}{k} D_{n-k}}{n!} = \frac{1}{k!}\sum_{j=0}^{n-k} \frac{(-1)^j}{j!}.$$

For $n = 52$:

- $P(52) = 1/52!$ (identity permutation).
- $P(51) = 0$ (no permutation has exactly 51 fixed points; the last one is forced).
- $P(50) = D_2/(50! \cdot 2!) = 1/(2 \cdot 50!)$.
- $P(49) = D_3 \cdot \binom{52}{49}/52! = 2 \cdot \binom{52}{3}/52! = 44\,200/52!$.
- General: $P(k) = D_{52-k}/[\,k!\,(52-k)!\,]$.

For large n and bounded k, $P(k) \to e^{-1}/k!$: the number of matches is asymptotically Poisson(1). The expected number of matches is exactly 1, regardless of n.

QUESTION 1.74. INTERVIEWER. n points on a circle are joined by all $\binom{n}{2}$ chords (no three concurrent inside). Into how many regions is the disc divided?

SHELDON. Apply Euler's formula $V - E + F = 2$ to the planar graph on the sphere.

Vertices. n on the boundary, plus one interior vertex per pair of crossing chords. Two chords cross inside iff their four endpoints are interleaved on the circle: exactly one interior crossing per 4-subset of the n boundary points.

$$V = n + \binom{n}{4}.$$

Edges. Boundary arcs: n. Each chord, when crossed internally c times, contributes $c + 1$ edge-segments. Summing over chords, the total internal contribution is $\binom{n}{2} + 2\binom{n}{4}$ (the factor 2 comes from each crossing being an endpoint for two chord-segments).

$$E = n + \binom{n}{2} + 2\binom{n}{4}.$$

Faces. Solving Euler $V - E + F = 2$:

$$F = 2 - V + E = 2 + \binom{n}{2} + \binom{n}{4}.$$

Subtracting the exterior region,

$$R(n) = 1 + \binom{n}{2} + \binom{n}{4}.$$

For $n = 1, \ldots, 6$: $R = 1, 2, 4, 8, 16, 31$. The tempting guess $R(n) = 2^{n-1}$ collapses at $n = 6$. The world, in such moments, is revealed to be $\binom{n}{4}$-deep rather than 2-deep.

Question 1.75. Interviewer. In the plane, $2n$ points have no three collinear; n are blue and n red. Prove the red points can be joined to the blue points by line segments in such a way that no two segments cross.

Sheldon. Consider all $n!$ perfect matchings between the reds and blues, and pick one of minimum total length. If any two segments $\overline{R_1B_1}$ and $\overline{R_2B_2}$ cross at a point P, by the triangle inequality applied to the quadrilateral,

$$|R_1B_2| + |R_2B_1| < |R_1B_1| + |R_2B_2|,$$

so swapping to the matching $R_1 \leftrightarrow B_2$, $R_2 \leftrightarrow B_1$ gives a strictly shorter total length, contradicting minimality. Hence the minimum-length matching has no crossings. Such "exchange" arguments are the mathematician's equivalent of reductio ad absurdum, except with feelings. I generally prefer them.

Question 1.76. Interviewer. Place the integers $1, 2, \dots, n^2$ in the cells of an $n \times n$ chessboard. Show that some pair of cells adjacent horizontally, vertically, or diagonally differs by at least $n+1$.

Sheldon. Define the *king distance* between two cells as the minimum number of king moves between them (each step moves to an orthogonally or diagonally adjacent cell). The king distance between any two cells of an $n \times n$ board is at most $n-1$.

Suppose for contradiction that every pair of adjacent cells differs by at most n. Then between the cell containing 1 and the cell containing n^2, along any king-path of length $\ell \le n-1$, the values change by at most n per step; so

$$n^2 - 1 \le n \cdot (n-1) = n^2 - n.$$

This gives $n - 1 \le 0$, absurd for $n \ge 2$. Hence some adjacent pair differs by at least $n+1$.

Question 1.77. Interviewer. n students stand in a field at pairwise distinct distances. Each throws a ball to the nearest student. Prove that some two students throw to each other.

Sheldon. Among the $\binom{n}{2}$ pairwise distances, consider the minimum, attained by a unique pair (A, B) (distinctness of distances guarantees uniqueness). Every other distance from A strictly exceeds $|AB|$, so A's nearest neighbour is B. Similarly, B's nearest neighbour is A. They throw to each other.

Question 1.78. Interviewer. A finite set of red and blue points has the property that every segment joining two same-coloured points contains a point of the other colour. Prove all points are collinear.

Sheldon. This is the coloured cousin of Sylvester–Gallai, and the honest route is through the *Motzkin–Rabin theorem*: *any two-*

colouring of a finite, non-collinear set of points admits a monochromatic connecting line, that is, a line through at least two of the points all of which share one colour. Granting that, the problem falls in two lines.

Suppose the points are not all collinear. By Motzkin–Rabin there is a connecting line ℓ all of whose configuration points are, say, red, and there are at least two of them. Walk along ℓ and take two *consecutive* red points B, C (no configuration point lies strictly between them on ℓ). The open segment BC contains no point of the set at all, in particular no blue point. But B and C are the same colour, so the hypothesis demands a point of the other colour inside BC. That contradiction forces all points onto one line.

The weight of the argument sits in Motzkin–Rabin itself, proved by exactly the extremal-distance descent that proves Sylvester–Gallai: assume every connecting line is bichromatic, take the point–line pair (P, ℓ) with $P \notin \ell$ minimising the positive distance $d(P, \ell)$, and the perpendicular foot together with a colour-changing pair on ℓ yields a connecting line strictly nearer to one of the points, contradicting minimality. I cite the theorem rather than re-run its case analysis on a borrowed whiteboard; the reduction above is the part the interviewer actually wanted to see.

Question 1.79. Interviewer. A *longevity chain* is a run of consecutive positive integers whose digit sums are never divisible by 9. What is the longest possible?

Sheldon. The digit sum of a positive integer n is congruent to $n \pmod 9$ (the classical "casting out nines"). Hence digit-sum divisible by 9 iff the number itself is divisible by 9. Among any nine consecutive integers exactly one is a multiple of 9, so a longevity chain has length at most 8. The run $1, 2, \ldots, 8$ attains this bound.

Answer: 8.

Question 1.80. Interviewer. A T-tetromino is four unit squares in a T-shape. A $2a \times 2b$ rectangle R is to be tiled by T-tetrominoes. For which integer a, b is this possible?

Sheldon. Area first: $|R| = 4ab$, a multiple of the tile's 4, so area is no obstruction. It is also no guarantee. The sharp fact, due to Walkup (1965), is that an $m \times n$ rectangle is tileable by T-tetrominoes *if and only if* $4 \mid m$ and $4 \mid n$ *both*. For $R = 2a \times 2b$ that requires $4 \mid 2a$ and $4 \mid 2b$, i.e. a and b are *both even*. The popular "one of the sides divisible by 4" is wrong: a 4×6 rectangle has a side divisible by 4 and is not tileable.

Sufficiency is a single picture. Four T-tetrominoes meet at the centre of a 4×4 square and tile it. If a and b are both even, $2a \times 2b = (4 \cdot \frac{a}{2}) \times (4 \cdot \frac{b}{2})$ is paved by a grid of such 4×4 blocks.

Both a, b odd is killed by colour. Chessboard the rectangle. A T-tetromino always covers three squares of one colour and one of the other; if k of the $\frac{mn}{4}$ tiles are "three-black," the black count is $3k + (\frac{mn}{4} - k) = 2k + \frac{mn}{4}$. A chessboard holds $\frac{mn}{2}$ black, forcing $2k = \frac{mn}{4}$, hence $8 \mid mn$. With a, b both odd, $mn = 4ab \equiv 4 \pmod 8$, so no tiling exists.

The mixed case (one of a, b even, the other odd, the 4×6 being the smallest) slips past this colour count yet is still untileable; excluding it is the substance of Walkup's theorem, which I cite rather than re-derive on a borrowed whiteboard. The answer stands: $2a \times 2b$ is tileable exactly when a and b are both even.

Question 1.81. Interviewer. Each cell of an 8×8 chessboard is coloured red, white, or blue. Are there more arrangements with an even number of red counters, or with an odd number?

Sheldon. For each cell independently, track the number of red counters by the generating function $(x + 2)$: the x marks "red," the 2 covers the two non-red choices. The total generating function over all 64 cells is $(x+2)^{64}$.

The number of arrangements with an even (resp. odd) count of reds is the sum of coefficients at even (resp. odd) powers of x, extracted by

$$\text{even} = \tfrac{(2+1)^{64}+(2-1)^{64}}{2} = \tfrac{3^{64}+1}{2}, \qquad \text{odd} = \tfrac{3^{64}-1}{2}.$$

The even count exceeds the odd count by exactly 1. The asymmetry is produced by the single all-non-red arrangement with zero reds, which has no odd counterpart.

Question 1.82. Interviewer. Prove that no cuboid has volume, surface area, and perimeter numerically equal.

Sheldon. Let the side lengths be $a, b, c > 0$, and set $s = a+b+c$, $q = ab + bc + ca$, $p = abc$. The perimeter is $4s$, the surface area $2q$, the volume p. Equality of all three gives $p = 2q$ and $2q = 4s$, hence

$$q = 2s, \qquad p = 4s.$$

Maclaurin's inequality for positive reals asserts $q^2 \geq 3sp$ (equivalently, $e_2^2 \geq 3e_1e_3$). Substituting the required equalities,

$$(2s)^2 \geq 3 \cdot s \cdot 4s \iff 4s^2 \geq 12s^2,$$

which fails for every $s > 0$. Hence no positive real cuboid satisfies the three equalities.

Question 1.83. Interviewer. On a 5×5 grid a token starts at the bottom-left. A move is one or two units rightward within the row, or to the leftmost square of the row above. Last player to move wins. Characterise winning and losing positions.

Sheldon. Label cells (r, c) with $r = 0$ at the bottom and $c = 0$ at the left. From (r, c), legal moves are $(r, c+1)$, $(r, c+2)$ (within the row), and $(r+1, 0)$ (if $r < 4$). Call a cell *losing* if the player to move from it loses, *winning* otherwise.

Work top-down. Row 4 has no "up" move, so it is the subtraction game $\{1,2\}$ with terminal at $c = 4$. Its losing positions are $c = 1$ and $c = 4$ (i.e. $c \equiv 1 \pmod 3$). For rows below, the up-move sends the opponent to $(r+1, 0)$, which is *winning* (column 0 is $0 \pmod 3$), so adding the up-move leaves the L-structure of each row unchanged. By induction, every row has the same losing columns $\{1, 4\}$.

Starting cell $(0, 0)$: winning. The first player wins, e.g. by moving to $(0, 1)$, a losing position for the opponent.

The 5×5 grid has $2 \times 5 = 10$ losing cells. On an $m \times n$ grid (with m columns), the losing columns are $c \equiv (m-1) \pmod 3$; their total is $n \cdot \lfloor (m+2)/3 \rfloor$.

Question 1.84. Interviewer. There are n sweets. Two players alternately remove at least one but no more than half of the remaining sweets. Whoever takes the last sweet loses. For which n does the second player have a winning strategy?

Sheldon. Write $L(n) = 1$ if the position with n sweets is losing for the player about to move, and $L(n) = 0$ if winning. Legal moves from n go to $n - k$ for $1 \le k \le \lfloor n/2 \rfloor$.

The base case needs a stated convention. At $n = 1$ the rule permits removing k with $1 \le k \le \lfloor 1/2 \rfloor = 0$, so there is *no legal move*. The loser is whoever is forced to take the last sweet; a player who cannot move has not taken it and so cannot lose on this turn. We therefore record $n = 1$ as a *win* for the player to move, $L(1) = 0$. (Equivalently: handing your opponent exactly one sweet wins, because they are stuck.) From here the recursion is the standard one, $L(n) = 1$ precisely when every legal move lands on a position with $L = 0$.

Compute iteratively: $L(2) = L(5) = L(11) = 1$ and $L = 0$ at $n \in \{1, 3, 4, 6, 7, 8, 9, 10\}$. The losing positions form the sequence $n = 2, 5, 11, 23, 47, \ldots$, each term $2 \cdot (\text{prev}) + 1$; explicitly,

$$n = 3 \cdot 2^k - 1, \qquad k = 0, 1, 2, \ldots$$

The second player wins iff the starting pile size is of this form. The doubling structure arises because from $n = 3 \cdot 2^k - 1$ the first player can remove at most $\lfloor n/2 \rfloor$, and the residual is bounded strictly above $3 \cdot 2^{k-1} - 1$, the nearest losing value below.

Question 1.85. Interviewer. Consider circles centred at $(\frac{1}{3}, \sqrt{2})$. Show that no such circle passes through two distinct lattice points; equivalently, every lattice point lies on exactly one circle of the family, and the circles *that pass through a lattice point* each contain exactly one.

Sheldon. The claim is really that the centre $(\frac{1}{3}, \sqrt{2})$ sits at a *distinct* distance from every lattice point (most circles in the family pass through no lattice point at all; the assertion concerns those that do). Let $(m, n) \in \mathbb{Z}^2$ lie on the circle of radius r. The defining equation reads

$$r^2 = \left(m - \tfrac{1}{3}\right)^2 + (n - \sqrt{2})^2 = m^2 + n^2 + \tfrac{1}{9} + 2 - \tfrac{2m}{3} - 2\sqrt{2}\,n.$$

Two lattice points (m_1, n_1) and (m_2, n_2) lie on the same circle iff their r^2 values match, which separates into rational and irrational parts:

$$m_1^2 + n_1^2 - \tfrac{2m_1}{3} = m_2^2 + n_2^2 - \tfrac{2m_2}{3}, \qquad -2\sqrt{2}\,(n_1 - n_2) = 0.$$

The second equation forces $n_1 = n_2$; substituting, the first gives $(m_1 - m_2)(m_1 + m_2 - \frac{2}{3}) = 0$. Since $m_1 + m_2$ is an integer and $\frac{2}{3}$ is not, the second factor never vanishes, so $m_1 = m_2$.

Hence distinct lattice points have distinct distances from the centre: exactly one lattice point per circle, and every lattice point is on some circle.

Question 1.86. Interviewer. Six ropes are in a bag. At each step, two free ends are picked at random, tied together, and returned to the bag. Continue until no free ends remain. What is the expected number of loops?

SHELDON. Let E_n denote the expected number of loops starting from n ropes (i.e. $2n$ free ends). The first step picks two ends out of $2n$: the probability they belong to the same rope, forming a loop and reducing the problem to $n-1$ ropes, is $\frac{1}{2n-1}$. With complementary probability, two distinct ropes are joined into one, also reducing to $n-1$ ropes but forming no loop. Hence

$$E_n = \tfrac{1}{2n-1} + E_{n-1}, \qquad E_0 = 0.$$

Telescoping,

$$E_n = \sum_{k=1}^{n} \tfrac{1}{2k-1} = 1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{2n-1}.$$

For $n = 6$: $E_6 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} = \frac{6508}{3465} \approx 1.878$.

QUESTION 1.87. INTERVIEWER. A thin rod is broken into three pieces at two uniformly chosen points. What is the probability that the three pieces form a triangle?

SHELDON. Normalise the rod to length 1 and let the two cut points be $U, V \sim \text{Uniform}(0,1)$ independently. Without loss of generality condition on $U < V$; the piece lengths are U, $V - U$, $1 - V$. The triangle inequality is equivalent to no piece exceeding $\frac{1}{2}$:

$$U < \tfrac{1}{2}, \quad V > \tfrac{1}{2}, \quad V - U < \tfrac{1}{2}.$$

Within the triangle $\{0 < U < V < 1\}$ (area $\frac{1}{2}$), the favourable region is $\{0 < U < \frac{1}{2}, \frac{1}{2} < V < U + \frac{1}{2}\}$, which has area $\int_0^{1/2} U\,dU = \frac{1}{8}$. Doubling for the symmetric case $V < U$ gives total favourable area $\frac{1}{4}$, and the probability is

$$P(\text{triangle}) = \tfrac{1}{4}.$$

QUESTION 1.88. INTERVIEWER. Show that $n!$ divides the product of any n consecutive positive integers.

SHELDON. For any integer $m \geq 1$,

$$m(m+1)(m+2)\cdots(m+n-1) = \frac{(m+n-1)!}{(m-1)!} = n!\cdot\binom{m+n-1}{n}.$$

The right-hand side is manifestly $n!$ times an integer, so $n! \mid m(m+1)\cdots(m+n-1)$.

QUESTION 1.89. INTERVIEWER. The sides of a triangle are in geometric progression with common ratio r. Prove that $\frac{2}{1+\sqrt{5}} < r < \frac{1+\sqrt{5}}{2}$.

SHELDON. WLOG scale so that the sides are $1, r, r^2$ with $r > 0$. The three triangle inequalities are:

$$1 + r > r^2, \qquad 1 + r^2 > r, \qquad r + r^2 > 1.$$

The middle inequality $r^2 - r + 1 > 0$ has discriminant $-3 < 0$ and is therefore automatic. The remaining two give

$$r^2 - r - 1 < 0 \iff r < \tfrac{1+\sqrt{5}}{2}, \quad r^2 + r - 1 > 0 \iff r > \tfrac{-1+\sqrt{5}}{2}.$$

Hence $\frac{2}{1+\sqrt{5}} < r < \frac{1+\sqrt{5}}{2}$, i.e. $\frac{1}{\varphi} < r < \varphi$ where φ is the golden ratio.

QUESTION 1.90. INTERVIEWER. How many integer solutions are there to $|x| + |y| \leq 100$?

SHELDON. Partition by the value $k = |x| + |y|$ for $k = 0, 1, \ldots, 100$. For $k = 0$ there is one solution, $(0, 0)$. For $k \geq 1$, the equation $|x| + |y| = k$ has exactly $4k$ lattice-point solutions (the rotated square has 4 vertices plus $k - 1$ points along each of its 4 open sides). Summing,

$$1 + \sum_{k=1}^{100} 4k = 1 + 4 \cdot \frac{100\cdot 101}{2} = 1 + 20\,200 = 20\,201.$$

QUESTION 1.91. INTERVIEWER. Are there coprime integer solutions to $x^2 + y^2 = 3z^2$? Integer solutions at all?

SHELDON. Squares modulo 3 lie in $\{0,1\}$, so $x^2+y^2 \pmod 3 \in \{0,1,2\}$ with $x^2+y^2 \equiv 0 \pmod 3$ only when $x \equiv y \equiv 0 \pmod 3$.

Suppose (x,y,z) is a nontrivial solution; then $3 \mid x^2+y^2 = 3z^2$, so $3 \mid x$ and $3 \mid y$. Substituting $x = 3x'$, $y = 3y'$: $9(x'^2+y'^2) = 3z^2$, whence $z^2 = 3(x'^2+y'^2)$, and hence $3 \mid z$. So the greatest common divisor of x,y,z is at least 3: no coprime solutions exist.

By infinite descent (dividing out common factors of 3), the only integer solution at all is $(x,y,z) = (0,0,0)$.

QUESTION 1.92. INTERVIEWER. Sketch $x^2 - ny^2 = 0$ for natural number n. Find all natural-number solutions for $n = 9$ and $n = 10$.

SHELDON. Factoring over the reals, $x^2 - ny^2 = 0 \iff (x - y\sqrt{n})(x + y\sqrt{n}) = 0$, so the locus is the pair of lines $x = \pm y\sqrt{n}$ through the origin.

For $n = 9$: $x = \pm 3y$. Natural-number solutions with $y \geq 1$ and $x \geq 1$: $(x,y) = (3y, y)$ for every $y \geq 1$, i.e. an infinite family.

For $n = 10$: $x = \pm y\sqrt{10}$ requires $\sqrt{10}$ rational, which it is not (since 10 is not a perfect square). Hence no natural-number solutions with $y \geq 1$. The only solution in the non-negative integers is $(0,0)$, which is not natural.

QUESTION 1.93. INTERVIEWER. How many natural-number triples (a,b,c) with $a < b < c$ satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$?

SHELDON. Since a is the smallest, $\frac{3}{a} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, so $a < 3$, i.e. $a \in \{1,2\}$. The case $a = 1$ forces $\frac{1}{b} + \frac{1}{c} = 0$, impossible for positive b,c. Hence $a = 2$, and we need $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ with $b > 2$.

Bound b: since $\frac{2}{b} > \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$, $b < 4$, so $b = 3$. Then $c = 6$, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ as required.

Unique solution: $(2, 3, 6)$. One triple.

Question 1.94. Interviewer. Find all positive integer solutions of $x^2 + y^2 = 2015$. Does $x^2 + 33y^2 = 555\,555\,555$ have any?

Sheldon. $2015 = 5 \cdot 13 \cdot 31$. Fermat's sum-of-two-squares theorem: $n \in \mathbb{N}$ is representable as a sum of two squares iff every prime factor of n of the form $4k + 3$ appears to an even power. Here $31 \equiv 3 \pmod 4$ and appears once, so no representation exists, and $x^2 + y^2 = 2015$ has no positive integer solutions.

For $x^2 + 33y^2 = 555\,555\,555$, descend by 3. First locate the right side in the 3-adic ladder: its digit sum is 45, so $9 \mid 555\,555\,555$, and $555\,555\,555 = 9 \cdot 61\,728\,395$, where $61\,728\,395$ has digit sum $41 \equiv 2 \pmod 3$, hence is *not* divisible by 3. Write $N = 555\,555\,555 = 9M$ with $M = 61\,728\,395 \equiv 2 \pmod 3$.

Now reduce the equation step by step.

- Modulo 3, since $33 \equiv 0$ and $N \equiv 0$, we get $x^2 \equiv 0 \pmod 3$, so $3 \mid x$. Put $x = 3x_1$: the equation becomes $9x_1^2 + 33y^2 = 9M$, i.e. $3x_1^2 + 11y^2 = 3M$.
- This forces $11y^2 \equiv 0 \pmod 3$, and as $11 \equiv 2$ is invertible, $3 \mid y$. (*This is the step that closes the descent: both x and y must carry the factor* 3.) Put $y = 3y_1$: then $3x_1^2 + 99y_1^2 = 3M$, i.e. $x_1^2 + 33y_1^2 = M$.

We have descended to $x_1^2 + 33y_1^2 = M$ with $M \equiv 2 \pmod 3$. Modulo 3 this reads $x_1^2 \equiv 2 \pmod 3$, impossible since squares are $\{0, 1\}$ modulo 3. No positive integer solutions.

Question 1.95. Interviewer. Find all positive integer solutions of $n^x + n^y = n^z$.

Sheldon. WLOG $x \le y$. Factor the left side: $n^x(1 + n^{y-x}) = n^z$.

If $y = x$: $2n^x = n^z$, so $2 = n^{z-x}$. This forces $n = 2$ and $z = x + 1$. The family is $(n, x, y, z) = (2, k, k, k + 1)$ for any $k \geq 1$.

If $y > x$: write $d = y - x \geq 1$. Then $1 + n^d = n^{z-x}$. But $n^d < 1 + n^d < n^{d+1}$ for $n \geq 2$, wedging $1 + n^d$ strictly between two consecutive powers of n: no integer exponent fits. Hence no solutions with $y > x$.

All solutions: $n = 2, x = y, z = x + 1$.

Question 1.96. Interviewer. Does $x^4 + 2x^3 + 3x^2 + 2x + 1 = 0$ have any real solutions?

Sheldon. The polynomial is a perfect square:

$$x^4 + 2x^3 + 3x^2 + 2x + 1 = (x^2 + x + 1)^2.$$

Verification by expansion is immediate. The inner quadratic $x^2 + x + 1$ has discriminant $1 - 4 = -3 < 0$, so it is never zero (and in fact positive) for real x. Hence the quartic is strictly positive for all real x, and the equation has no real solutions.

Question 1.97. Interviewer. Find all solutions of (a) $|x| + |x - 1| = 0$, (b) $|27x^2 - 48| + |6x^2 - 5x - 4| = 0$, (c) $|\sin 2x| + |\cos(x/2)| = 0$.

Sheldon. Each equation is a sum of non-negative quantities equal to zero, so *each summand must vanish independently*.

(a) $|x| = 0 \iff x = 0$ and $|x - 1| = 0 \iff x = 1$; both must hold, which is impossible. No solutions.

(b) $27x^2 = 48 \iff x^2 = \frac{16}{9} \iff x = \pm\frac{4}{3}$. Check $6x^2 - 5x - 4$: at $x = \frac{4}{3}$, $6 \cdot \frac{16}{9} - \frac{20}{3} - 4 = 0$; at $x = -\frac{4}{3}$, $6 \cdot \frac{16}{9} + \frac{20}{3} - 4 = \frac{40}{3} \neq 0$. Unique solution $x = \frac{4}{3}$.

(c) $\sin 2x = 0 \iff x = k\pi/2$. $\cos(x/2) = 0 \iff x = (2m+1)\pi$. Intersection: $x = (2m+1)\pi$, which in the first form has $k = 2(2m + 1)$, all integers. So solutions are $x = (2m+1)\pi$ for $m \in \mathbb{Z}$.

Question 1.98. Interviewer. Positive reals a, b, c satisfy $abc = 1$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3$, and $a + b + c = 3$. Show that at least one of them is 1 and deduce all solutions.

Sheldon. Using $abc = 1$,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ca+ab}{abc} = ab + bc + ca = 3.$$

So a, b, c are roots of

$$t^3 - (a+b+c)t^2 + (ab+bc+ca)t - abc = t^3 - 3t^2 + 3t - 1 = (t-1)^3.$$

The only root is $t = 1$, hence $a = b = c = 1$. Not only is one of them 1: *all three* are.

Question 1.99. Interviewer. Prove that the only integer solution of $x^2 + y^2 + z^2 = 2xyz$ is $x = y = z = 0$.

Sheldon. Work modulo 4. Squares modulo 4 are $\{0, 1\}$, so $x^2 + y^2 + z^2 \pmod 4$ depends only on how many of x, y, z are odd, giving $0, 1, 2$, or 3 mod 4.

Case k of $\{x, y, z\}$ are odd, $k = 0, 1, 2, 3$:

- $k = 3$: LHS $\equiv 3$, RHS $= 2xyz \equiv 2 \pmod 4$. Mismatch.
- $k = 2$: LHS $\equiv 2$, RHS has xyz even so RHS $\equiv 0 \pmod 4$. Mismatch.
- $k = 1$: LHS $\equiv 1$, RHS $\equiv 0 \pmod 4$. Mismatch.
- $k = 0$: all three even.

So any solution has all of x, y, z even. Write $x = 2x'$, $y = 2y'$, $z = 2z'$: the equation becomes $4(x'^2 + y'^2 + z'^2) = 16x'y'z'$, i.e. $x'^2 + y'^2 + z'^2 = 4x'y'z'$. A similar mod-4 argument on this equation again forces x', y', z' all even. Iterating, $2^n \mid x, y, z$ for every n, so $x = y = z = 0$. Classical infinite descent; Fermat would approve.

Question 1.100. Interviewer. Show that no three real numbers a, b, c satisfy both $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ and $a + b + c = 0$.

Sheldon. The first equation, cleared of denominators, gives $ab+bc+ca = 0$; the second, $a+b+c = 0$. Substitute $c = -(a+b)$ into $ab + bc + ca = 0$:

$$ab + c(a + b) = ab - (a + b)^2 = -(a^2 + ab + b^2) = 0.$$

So $a^2 + ab + b^2 = 0$. Viewing this as a quadratic in a, the discriminant is $b^2 - 4b^2 = -3b^2 \leq 0$, vanishing only at $b = 0$. Non-trivial real solutions require a negative discriminant-free case, which occurs only at $a = b = 0$, forcing $c = 0$; but then $\frac{1}{a}$ is undefined.

Hence no real triple (a, b, c) with all three reciprocals defined satisfies both equations.

Question 1.101. Interviewer. If x, y are positive integers, find all solutions of $2xy - 4x^2 + 12x - 5y = 11$.

Sheldon. Solve for y: $y(2x - 5) = 4x^2 - 12x + 11$, so $y = \dfrac{4x^2 - 12x + 11}{2x - 5}$. Polynomial division gives

$$4x^2 - 12x + 11 = (2x - 5)(2x - 1) + 6, \qquad y = 2x - 1 + \frac{6}{2x - 5}.$$

For integer y, $(2x-5) \mid 6$. The positive divisors of 6 are $1, 2, 3, 6$; only the odd ones give integer $x \geq 3$. So $2x - 5 \in \{1, 3\}$, giving $(x, y) = (3, 11)$ and $(4, 9)$. Negative divisors yield non-positive y.

The solutions are $(3, 11)$ and $(4, 9)$.

Question 1.102. Interviewer. Find all right-angled triangles with integer sides whose perimeter equals their area.

Sheldon. Let the legs be a, b and hypotenuse $c = \sqrt{a^2 + b^2}$. The condition is $a + b + c = \frac{1}{2}ab$, so $c = \frac{1}{2}ab - a - b$. Squaring and using $c^2 = a^2 + b^2$:

$$a^2 + b^2 = \tfrac{1}{4}a^2b^2 - ab(a + b) + (a + b)^2.$$

Cancelling $a^2 + b^2$ from both sides and dividing by ab,

$$\tfrac{1}{4}ab - (a+b) + 2 = 0 \iff (a-4)(b-4) = 8.$$

Positive integer factorisations: $(a-4, b-4) \in \{(1,8),(2,4),(4,2),(8,1)\}$, giving (up to swap) $(a,b) = (5,12)$ with $c = 13$, and $(a,b) = (6,8)$ with $c = 10$.

Two triangles: $(5,12,13)$ and $(6,8,10)$. Both have perimeter = area = 30 and 24 respectively.

Question 1.103. Interviewer. Prove that $n^2(n^2-1)(n^2-4)$ is divisible by 360 for every natural n.

Sheldon. Factor: $n^2(n^2-1)(n^2-4) = n \cdot [(n-2)(n-1)\,n\,(n+1)(n+2)]$. The bracketed term is the product of five consecutive integers, divisible by $5! = 120$. We need an extra factor of 3 to reach $360 = 120 \cdot 3$.

Among $(n-2, n-1, n, n+1, n+2)$, the residues mod 3 comprise either one or two multiples of 3. If two, the bracket is divisible by 9; combined with the 120 factor, the total has $9 \cdot 5 \cdot 8 = 360$. If only one multiple, that multiple is necessarily n itself (the other four cover residues $\{\pm1, \pm2\} \pmod 3$); in this case $n \equiv 0 \pmod 3$, so the leading n contributes the second factor of 3, again giving 360.

Question 1.104. Interviewer. Find the last two digits of 99^n.

Sheldon. $99 \equiv -1 \pmod{100}$, so $99^n \equiv (-1)^n \pmod{100}$. Hence the last two digits are 01 for even n and 99 for odd n.

Question 1.105. Interviewer. Solve $5^x = 4^x + 3^x$. Are there other solutions?

Sheldon. $x = 2$ works: $25 = 16 + 9$. Divide both sides by 5^x:

$$1 = \left(\tfrac{4}{5}\right)^x + \left(\tfrac{3}{5}\right)^x.$$

The right side is a strictly decreasing function of x (each base is less than 1), continuous, with limits 2 at $x = 0^+$ and 0 at $x \to \infty$. By the intermediate value theorem and monotonicity, the equation $f(x) = 1$ has exactly one real solution. Hence $x = 2$ is the unique solution.

The Pythagorean triple $(3, 4, 5)$ is doing the work, and one feels obliged to remark that $n = 2$ is the *largest* exponent for which $a^n + b^n = c^n$ has non-trivial positive-integer solutions: $n = 1$ has them trivially $(a + b = c)$, $n = 2$ has the Pythagorean triples, and for every $n \geq 3$ there are none, the observation due to Fermat in the margin and to Wiles in 1994.

Question 1.106. Interviewer. Prove (a) $8 \mid n^2 - 1$ for n odd; (b) $6 \mid n^5 - n$; (c) $30 \mid n^5 - n$ for all natural n.

Sheldon. (a) For odd $n = 2k + 1$: $n^2 - 1 = 4k(k + 1)$. Since $k(k+1)$ is the product of two consecutive integers, one is even, so $k(k + 1)$ is even, hence $n^2 - 1 = 8 \cdot \frac{k(k+1)}{2}$.

(b) $n^5 - n = n(n^4 - 1) = n(n - 1)(n + 1)(n^2 + 1)$. The factor $n(n - 1)(n + 1)$ is the product of three consecutive integers, divisible by $3! = 6$.

(c) By Fermat's little theorem, $n^5 \equiv n \pmod 5$, so $5 \mid n^5 - n$. Combined with (b) and $\gcd(5, 6) = 1$, $30 \mid n^5 - n$.

Question 1.107. Interviewer. Simplify $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (2n-1)^2 - (2n)^2$. Hence find $21^2 - 22^2 + 23^2 - 24^2 + \cdots + 39^2 - 40^2$.

Sheldon. Each pair telescopes via difference of squares:

$$(2k - 1)^2 - (2k)^2 = (2k - 1 - 2k)(2k - 1 + 2k) = -(4k - 1).$$

Summing $k = 1$ to n:

$$S_n = -\sum_{k=1}^{n}(4k - 1) = -[2n(n + 1) - n] = -n(2n + 1).$$

For the second sum, k ranges from 11 to 20 (so $2k-1 = 21,\dots,39$ and $2k = 22,\dots,40$):

$$\sum_{k=11}^{20} -(4k-1) = -[4 \cdot \tfrac{(11+20)\cdot 10}{2} - 10] = -(620-10) = -610.$$

QUESTION 1.108. INTERVIEWER. Construct a counterexample to: "every square number, in decimal, has at most 1000 digits that are not 0 or 1."

SHELDON. Take $N = \underbrace{99\dots9}_{1001 \text{ nines}} = 10^{1001} - 1$. Its square is

$$N^2 = (10^{1001}-1)^2 = 10^{2002} - 2\cdot 10^{1001} + 1 = \underbrace{99\dots9}_{1000}\,8\,\underbrace{00\dots0}_{1000}\,1,$$

which has 1001 digits not in $\{0,1\}$ (namely the 1000 nines and one 8). This exceeds the threshold of 1000, so the statement is false.

QUESTION 1.109. INTERVIEWER. Prove that $4^n - 1$ is divisible by 3 for every natural n.

SHELDON. $4 \equiv 1 \pmod 3$, so $4^n \equiv 1^n \equiv 1 \pmod 3$, hence $3 \mid 4^n - 1$.

QUESTION 1.110. INTERVIEWER. A natural number from 1 to 10^6 is selected at random. What is the probability its cube ends in 11?

SHELDON. We need $n^3 \equiv 11 \pmod{100}$. By CRT split into modulo 4 and 25.

Modulo 4. $11 \equiv 3$. Cubes mod 4: $0, 1, 0, 3$, so $n \equiv 3 \pmod 4$.

Modulo 25. Direct enumeration of $n^3 \pmod{25}$ for $n = 0,\dots,24$ produces 11 only at $n = 21$.

CRT-combining $n \equiv 3 \pmod 4$ and $n \equiv 21 \pmod{25}$: the unique solution mod 100 is $n \equiv 71$. Verification: $71^3 = 357\,911$, which indeed ends in 11.

The interval $[1, 10^6]$ contains exactly 10^4 values congruent to 71 mod 100. Probability:

$$\frac{1}{100}.$$

Question 1.111. Interviewer. Given $8 < \pi^2 < 10$, show $\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_5 \pi} > 2$ and $\dfrac{1}{\log_2 \pi} + \dfrac{1}{\log_\pi 2} > 2$.

Sheldon. Use the change-of-base $\frac{1}{\log_b a} = \log_a b$.

First inequality. $\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} = \log_\pi 2 + \log_\pi 5 = \log_\pi 10$. Since $\pi > 1$ and $10 > \pi^2$, $\log_\pi 10 > 2$.

Second inequality. Let $t = \log_2 \pi > 1$. The expression equals $\frac{1}{t} + t$. By AM–GM (or $(x-1)^2 \geq 0$ at $x = \sqrt{t}$), $t + \frac{1}{t} \geq 2$ with equality iff $t = 1$, i.e. $\pi = 2$. Since $\pi \neq 2$, the inequality is strict.

Question 1.112. Interviewer. Prove there are infinitely many primes.

Sheldon. *Euclid, c. 300 BCE.* Suppose only finitely many primes exist: $p_1, p_2, \ldots, p_k$. Form $N = p_1 p_2 \cdots p_k + 1$. For each i, dividing N by p_i leaves remainder 1, so no p_i divides N. Hence either N itself is a prime not in the list, or N has a prime factor not in the list. Either way the list was incomplete: contradiction.

Question 1.113. Interviewer. Prove there are infinitely many primes of the form $4n + 3$.

Sheldon. Suppose only finitely many: $q_1, \ldots, q_k$. Form $N = 4q_1 q_2 \cdots q_k - 1$, so $N \equiv 3 \pmod 4$ and N is odd, hence has only odd prime factors.

A product of primes each $\equiv 1 \pmod 4$ is itself $\equiv 1 \pmod 4$. Since $N \equiv 3 \pmod 4$, at least one prime factor p of N satisfies $p \equiv 3 \pmod 4$. But $p \neq q_i$ for any i, since $\gcd(N, q_i) \mid \gcd(N, q_1 \cdots q_k) = \gcd(N, \frac{N+1}{4}) = 1$. So p is a new prime $\equiv 3 \pmod 4$. Contradiction. A small Dirichlet-flavoured cousin; the full Dirichlet theorem on primes in arithmetic progressions $a + nq$ (with $\gcd(a, q) = 1$) requires analytic methods I shall not voluntarily inflict on a maths interview.

Question 1.114. Interviewer. Is $\log_2 3$ rational?

Sheldon. Suppose $\log_2 3 = \frac{p}{q}$ in lowest terms with $p, q > 0$. Then $2^{p/q} = 3$, so $2^p = 3^q$. The left side is a power of 2 (factorisation contains only the prime 2); the right is a power of 3. By unique factorisation, both must equal 1, requiring $p = q = 0$. But $\frac{p}{q}$ is then undefined. Contradiction.

Hence $\log_2 3$ is irrational.

Question 1.115. Interviewer. Prove that $14^n + 11$ is never prime.

Sheldon. Two-case mod analysis.

n even. $14 \equiv -1 \pmod 3$, so $14^n \equiv 1 \pmod 3$ for even n. Thus $14^n + 11 \equiv 1 + 2 \equiv 0 \pmod 3$, and the value is > 3, hence composite.

n odd. $14 \equiv -1 \pmod 5$, so $14^n \equiv -1 \pmod 5$ for odd n. Thus $14^n + 11 \equiv -1 + 1 \equiv 0 \pmod 5$, and the value is > 5, hence composite.

In every case $14^n + 11$ has a non-trivial divisor in $\{3, 5\}$, so is never prime.

Question 1.116. Interviewer. Suppose $a^n - 1$ is prime, $n \geq 2$. Show $a = 2$ and that n must be prime. Comment on $2^n + 1$ (Fermat numbers).

Sheldon. The factorisation $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \cdots + 1)$ exhibits two factors. If $a^n - 1$ is prime, one factor must be 1. The second factor is at least $n \geq 2$, so we need $a - 1 = 1$, i.e. $a = 2$.

If $n = rs$ with $r, s > 1$, then $2^n - 1 = (2^r)^s - 1$ is divisible by $2^r - 1$, which lies in $(1, 2^n - 1)$: composite. Hence for $2^n - 1$ to be prime, n must itself be prime. (The converse fails: $2^{11} - 1 = 23 \cdot 89$ is composite.) These are Mersenne primes.

Fermat numbers $2^n + 1$. The factorisation $a^m + 1 = (a+1)(a^{m-1} - a^{m-2} + \cdots + 1)$ holds when m is odd. So if n has an odd factor $m > 1$, then $2^n + 1 = (2^{n/m})^m + 1$ factors and is composite. For $2^n + 1$ to be prime, n must have no odd factor greater than 1: n is a power of 2. These are Fermat numbers $F_k = 2^{2^k} + 1$. Fermat conjectured all are prime; $F_5 = 4\,294\,967\,297 = 641 \cdot 6\,700\,417$ disabused him.

Question 1.117. Interviewer. Find all primes p such that $2p - 1$ and $2p + 1$ are also prime.

Sheldon. For $p > 3$, $p \not\equiv 0 \pmod 3$, so $p \equiv 1$ or $2 \pmod 3$, hence $2p \equiv 2$ or $1 \pmod 3$. In either case one of $\{2p-1, 2p+1\}$ is divisible by 3 and exceeds 3, hence composite.

So $p \leq 3$. Check: $p = 2$ gives $(3, 5)$, both prime. $p = 3$ gives $(5, 7)$, both prime.

All solutions: $p \in \{2, 3\}$.

Question 1.118. Interviewer. Show that $3 < \pi < 4$.

Sheldon. Bound π between the area of an inscribed and a circumscribed polygon for the unit circle.

Lower bound. A regular 12-gon inscribed in the unit circle decomposes into 12 congruent isosceles triangles, each with two sides 1 and apex angle $\frac{\pi}{6}$. Each triangle's area is $\frac{1}{2} \sin \frac{\pi}{6} = \frac{1}{4}$, so the polygon's area is $12 \cdot \frac{1}{4} = 3$. The polygon lies inside the circle, so $3 < \pi$.

Upper bound. A square circumscribed about the unit circle has side 2 and area 4, and contains the circle, so $\pi < 4$.

Hence $3 < \pi < 4$.

Question 1.119. Interviewer. If the ratio of consecutive Fibonacci numbers tends to a limit, what must that limit be?

Sheldon. Suppose $\lim_{n\to\infty} F_{n+1}/F_n = L$. Divide the recurrence $F_{n+1} = F_n + F_{n-1}$ by F_n:

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} \xrightarrow{n\to\infty} 1 + \frac{1}{L}.$$

So $L = 1 + \frac{1}{L}$, i.e. $L^2 - L - 1 = 0$, with positive root

$$L = \frac{1+\sqrt{5}}{2} = \varphi,$$

the golden ratio. Established by Kepler in 1611, popularised by Fibonacci 400 years earlier without his knowledge of the limit.

Question 1.120. Interviewer. Find the exact value of $\cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 89^\circ$.

Sheldon. Pair angle k with $90 - k$ for $k = 1, \dots, 44$:

$$\cos^2 k^\circ + \cos^2(90-k)^\circ = \cos^2 k^\circ + \sin^2 k^\circ = 1.$$

That accounts for 44 pairs summing to 44. The unpaired middle term, $k = 45$, contributes $\cos^2 45^\circ = \frac{1}{2}$. Total

$$\sum_{k=1}^{89} \cos^2 k^\circ = 44 + \frac{1}{2} = \frac{89}{2}.$$

Question 1.121. Interviewer. If a natural number n has N digits, how many digits can n^2 have? What about n^n? Write a formula for the number of digits of n.

Sheldon. The number of decimal digits of a positive integer m is $\lfloor \log_{10} m \rfloor + 1$.

Number of digits of n^2. If n has N digits, $10^{N-1} \le n < 10^N$, so $10^{2N-2} \le n^2 < 10^{2N}$. Hence n^2 has either $2N-1$ or $2N$ digits.

Number of digits of n^n. $\log_{10}(n^n) = n\log_{10} n$, so n^n has $\lfloor n\log_{10} n\rfloor + 1$ digits.

General formula. $\#\text{digits}(n) = \lfloor \log_{10} n\rfloor + 1$. The trick of writing it without the floor by other means is unavailable; integer-counts are intrinsically discrete and the floor is doing the right job.

Question 1.122. Interviewer. Given $\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}$, find $\sum_{r=1}^{\infty} \frac{1}{(2r-1)^2}$.

Sheldon. Split the harmonic series of squares into odd and even index sets:

$$\sum_{r\ge 1} \frac{1}{r^2} = \sum_{r\ge 1} \frac{1}{(2r-1)^2} + \sum_{r\ge 1} \frac{1}{(2r)^2}.$$

The even sum factors: $\sum \frac{1}{(2r)^2} = \frac{1}{4}\sum \frac{1}{r^2} = \frac{\pi^2}{24}$. Subtracting,

$$\sum_{r\ge 1} \frac{1}{(2r-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

Question 1.123. Interviewer. Prove that if a, b, c are all odd, then $ax^2 + bx + c = 0$ has no rational roots.

Sheldon. Suppose for contradiction $\frac{p}{q}$ is a rational root in lowest terms. Multiplying through by q^2, $ap^2 + bpq + cq^2 = 0$. Examine modulo 2 in cases:

- Both p and q odd: $ap^2+bpq+cq^2 \equiv 1+1+1 \equiv 1 \pmod 2$.
- p odd, q even: $\equiv 1+0+0 \equiv 1 \pmod 2$.
- p even, q odd: $\equiv 0+0+1 \equiv 1 \pmod 2$.
- Both even: forbidden by lowest-terms.

In every admissible case the left side is odd, hence non-zero. Contradiction.

Question 1.124. Interviewer. Is $\tan 1^\circ$ irrational? What about $\cos 1^\circ$?

Sheldon. Both are irrational.

$\tan 1^\circ$. Suppose $\tan 1^\circ$ is rational. The double-angle formula $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$ takes rationals to rationals, so $\tan 2^\circ, \tan 4^\circ, \tan 8^\circ, \ldots$ are rational. The angle-sum formula $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ then makes $\tan k^\circ$ rational for every positive integer k. But $\tan 60^\circ = \sqrt{3}$ is irrational. Contradiction.

$\cos 1^\circ$. By Niven's theorem, the only rational values of $\cos\theta$ when θ is a rational multiple of degrees are $0, \pm\frac{1}{2}, \pm 1$. Since $\cos 1^\circ$ is none of these, it is irrational.

Question 1.125. Interviewer. Find $\lim_{n\to\infty}\left(\frac{1^2}{n^3}+\frac{2^2}{n^3}+\cdots+\frac{n^2}{n^3}\right)$.

Sheldon. This is a Riemann sum for $\int_0^1 x^2\,dx$ with partition $x_k = k/n$ and width $1/n$:

$$\sum_{k=1}^{n}\frac{k^2}{n^3} = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 \xrightarrow{n\to\infty} \int_0^1 x^2\,dx = \frac{1}{3}.$$

Verification by closed form: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, so the sum equals $\frac{(n+1)(2n+1)}{6n^2} \to \frac{2}{6} = \frac{1}{3}$.

Question 1.126. Interviewer. Evaluate $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

Sheldon. Factor: $1-\frac{1}{n^2} = \frac{(n-1)(n+1)}{n^2}$. The partial product telescopes:

$$\prod_{n=2}^{N}\frac{(n-1)(n+1)}{n^2} = \left(\prod_{n=2}^{N}\frac{n-1}{n}\right)\left(\prod_{n=2}^{N}\frac{n+1}{n}\right) = \frac{1}{N}\cdot\frac{N+1}{2} = \frac{N+1}{2N} \xrightarrow{N\to\infty} \frac{1}{2}.$$

Question 1.127. Interviewer. Conjecture and prove a formula for $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n!$.

Sheldon. Note the identity $k \cdot k! = (k+1)! - k!$. Telescoping,

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n} \left[(k+1)! - k! \right] = (n+1)! - 1.$$

Question 1.128. Interviewer. Is 1234567891011 a square? Is 246810121416182022224?

Sheldon. Squares modulo 9 lie in $\{0,1,4,7\}$. Since the digit sum of n is congruent to $n \pmod 9$:

$N_1 = 1234567891011$. Digit sum $= 1+2+3+4+5+6+7+8+9+1+0+1+1 = 48 \equiv 3 \pmod 9$. Since $3 \notin \{0,1,4,7\}$, N_1 is not a square.

$N_2 = 246810121416182022224$. Digit sum $= 2+4+6+8+1+0+1+2+1+4+1+6+1+8+2+0+2+2+2+4 = 57 \equiv 3 \pmod 9$. Same residue, same conclusion: not a square.

Both fail by the same casting-out-nines test.

Question 1.129. Interviewer. If $f(x,y) = k \cdot f(y,x)$ for a non-identically-zero function f, what are the possible values of k?

Sheldon. Apply the relation twice: $f(x,y) = k \cdot f(y,x) = k \cdot [k \cdot f(x,y)] = k^2 f(x,y)$. So $(k^2-1)f(x,y) = 0$ for all (x,y). Since f is not identically zero, $k^2 = 1$, giving $k \in \{+1,-1\}$.

The two cases correspond to symmetric $(k=+1)$ and antisymmetric $(k=-1)$ functions; both are well-populated families.

Question 1.130. Interviewer. For $h(x) = x^3 + ax$ with a constant, when does an inverse exist for all x?

Sheldon. h is a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$ iff it is strictly monotonic. Differentiate: $h'(x) = 3x^2 + a$. This is positive for all x iff $a \geq 0$ (when $a = 0$, $h'(x) = 3x^2 \geq 0$ with equality only at the isolated point $x = 0$, which still leaves h strictly increasing).

For $a < 0$, h' has two real zeros and h has a local max and a local min, so it is not injective.

Inverse exists iff $a \geq 0$.

Question 1.131. Interviewer. Suppose $f(0) = 0$ and, for $x \neq 0$, $0 < \frac{f(x)}{x} < 1$. (a) Show $-\frac{1}{2} < \int_{-1}^{1} f\,dx < \frac{1}{2}$. (b) How does the bound change if $0 < \frac{f(x)}{x^2} < 1$?

Sheldon. (a) The hypothesis gives $0 < f(x) < x$ on $(0, \infty)$ and $x < f(x) < 0$ on $(-\infty, 0)$. Integrating on $[0,1]$, $0 < \int_0^1 f\,dx < \int_0^1 x\,dx = \frac{1}{2}$. Integrating on $[-1,0]$, $-\frac{1}{2} = \int_{-1}^{0} x\,dx < \int_{-1}^{0} f\,dx < 0$. Adding,

$$-\tfrac{1}{2} < \int_{-1}^{1} f\,dx < \tfrac{1}{2}.$$

(b) Now $0 < f(x) < x^2$ for all $x \neq 0$, in particular f is positive everywhere except 0. Hence $0 < \int_{-1}^{1} f\,dx < \int_{-1}^{1} x^2\,dx = \frac{2}{3}$. The new bounds are $0 < \int_{-1}^{1} f\,dx < \frac{2}{3}$.

Question 1.132. Interviewer. Show $\cos n\theta = f_n(\cos\theta)$ for polynomials satisfying $f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x)$. Find all roots of $f_2(x) + f_3(x) = 0$, expressed as $\cos\phi$.

Sheldon. The polynomials are the Chebyshev polynomials of the first kind. Verify the recursion via the product-to-sum identity:

$$2\cos\theta\cos n\theta = \cos(n+1)\theta + \cos(n-1)\theta,$$

so writing $\cos n\theta = f_n(\cos\theta)$ gives $f_{n+1}(x) = 2xf_n(x) - f_{n-1}(x)$ as required, with $f_0 = 1, f_1 = x$.

For $f_2 + f_3 = 0$ apply the sum-to-product formula in θ:

$$\cos 2\theta + \cos 3\theta = 2\cos\tfrac{5\theta}{2}\cos\tfrac{\theta}{2} = 0.$$

Solutions on $[0, 2\pi)$: $\frac{5\theta}{2} = \frac{\pi}{2} + k\pi$ gives $\theta \in \{\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}\}$; $\frac{\theta}{2} = \frac{\pi}{2}$ gives $\theta = \pi$. Distinct cos values are $\cos\frac{\pi}{5}$, $\cos\frac{3\pi}{5}$, and

$\cos \pi = -1$ (the others coincide with these by even symmetry of cos). The cubic $f_2 + f_3 = 4x^3 + 2x^2 - 3x - 1$ therefore has roots

$$x = \cos \frac{\pi}{5}, \quad \cos \frac{3\pi}{5}, \quad \cos \pi = -1.$$

Question 1.133. Interviewer. Find conditions on a, b, c, d for $y = ax^3 + bx^2 + cx + d$ to have a local maximum and a local minimum, and show the inflection point lies midway between them.

Sheldon. $y'(x) = 3ax^2 + 2bx + c$. For two distinct turning points the discriminant must be positive: $4b^2 - 12ac > 0$, i.e. $b^2 > 3ac$ (and $a \neq 0$).

The two roots x_1, x_2 satisfy $x_1 + x_2 = -\frac{2b}{3a}$ by Vieta's. The second derivative $y''(x) = 6ax + 2b$ vanishes at

$$x = -\frac{b}{3a} = \frac{x_1 + x_2}{2},$$

the midpoint.

Question 1.134. Interviewer. Find all real-valued functions satisfying $f(x+y)^2 = f(x)^2 + f(y)^2$.

Sheldon. Set $x = y = 0$: $f(0)^2 = 2f(0)^2$, so $f(0) = 0$. Set $y = -x$: $f(0)^2 = f(x)^2 + f(-x)^2$, i.e. $0 = f(x)^2 + f(-x)^2$. Two non-negative quantities sum to zero only if both are zero, so $f(x) = 0$ for all x.

Only solution: the zero function. The squared-additive constraint is more rigid than ordinary Cauchy additivity, which has wild solutions; squaring forces collapse.

Question 1.135. Interviewer. What are the domain and range of $f(x) = \ln x, f(f(x)), f(f(f(x)))$? What about $f^n(x)$?

Sheldon. Each composition demands its argument to be positive. Iterate the constraint:

- $f(x) = \ln x$: domain $(0, \infty)$, range $\mathbb{R}$.
- $f \circ f(x) = \ln \ln x$: requires $\ln x > 0 \iff x > 1$. Domain $(1, \infty)$, range $\mathbb{R}$.
- $f \circ f \circ f(x)$: requires $\ln \ln x > 0 \iff \ln x > 1 \iff x > e$. Domain (e, ∞), range $\mathbb{R}$.

For f^n, the domain is $(\, e \uparrow\uparrow (n-1)\, , \infty)$ where $e \uparrow\uparrow k$ is the tower $e^{e^{\cdot^{\cdot^{\cdot^{e}}}}}$ of height k. Specifically, the threshold sequence $T_0 = 0$, $T_1 = 1$, $T_2 = e$, $T_3 = e^e$, satisfies $T_{n+1} = e^{T_n}$. Range remains all of $\mathbb{R}$ for every n.

Question 1.136. Interviewer. Find the value of $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$.

Sheldon. Let x denote the value (assumed convergent). Then $x = 1 + \frac{1}{x}$, i.e. $x^2 - x - 1 = 0$, with positive root

$$x = \tfrac{1+\sqrt{5}}{2} = \varphi,$$

the golden ratio. The continued fraction is the slowest-converging of all simple continued fractions: every partial quotient is 1.

Question 1.137. Interviewer. Find the smallest $a > 1$ such that $\dfrac{a + \sin x}{a + \sin y} \leq e^{y-x}$ for all $x \leq y$.

Sheldon. Take logarithms: the inequality becomes $\ln(a + \sin x) - \ln(a + \sin y) \leq y - x$, equivalently $\ln(a + \sin t)$ is 1-Lipschitz from above as t increases (and -1-Lipschitz from below as t decreases). Differentiating,

$$\frac{d}{dt} \ln(a + \sin t) = \frac{\cos t}{a + \sin t}.$$

The Lipschitz condition demands $\frac{\cos t}{a + \sin t} \geq -1$ for all t, i.e. $\cos t + \sin t \geq -a$. Since $\min_t(\cos t + \sin t) = -\sqrt{2}$, the condition holds iff $a \geq \sqrt{2}$. The smallest such $a > 1$ is

$$a = \sqrt{2}.$$

Question 1.138. Interviewer. A non-constant f satisfies $f(x+y) = f(x)f(y)$. Show $f(n) = k^n$ for integers n with $k = f(1)$, and that the same holds for rationals with $k > 0$.

Sheldon. Set $y = 0$: $f(x) = f(x)f(0)$, so either $f \equiv 0$ (excluded) or $f(0) = 1$. Set $y = -x$: $1 = f(x)f(-x)$, so f is non-vanishing.

By induction, $f(n) = f(1)^n = k^n$ for $n \in \mathbb{N}$, and $f(-n) = 1/k^n = k^{-n}$. So $f(n) = k^n$ on $\mathbb{Z}$.

For rational $q = p/n$ with $n > 0$: $f(p) = k^p$ and $f(p) = f(n \cdot q) = f(q)^n$, so $f(q)^n = k^p$, giving $f(q) = k^{p/n} = k^q$ (taking the positive n-th root, justified by positivity).

Positivity. For any x, $f(x) = f(x/2 + x/2) = f(x/2)^2 \geq 0$; and f is non-vanishing, so $f(x) > 0$. In particular, $k = f(1) > 0$.

(The continuous extension to all reals is $f(x) = k^x = e^{x \ln k}$; without continuity, Cauchy's pathological solutions intervene.)

Question 1.139. Interviewer. Where in the plane is $\sin^2 x + \cos^2 y = 1$?

Sheldon. Using $\cos^2 y = 1 - \sin^2 y$, the condition becomes $\sin^2 x = \sin^2 y$, i.e. $|\sin x| = |\sin y|$, i.e. $\sin x = \pm \sin y$.

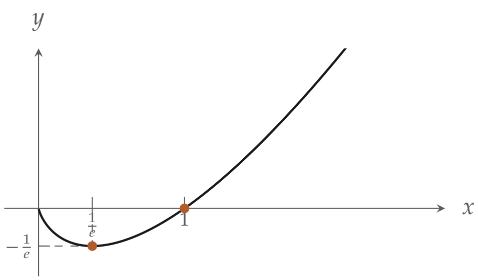
Case $\sin x = \sin y$: $x = y + 2k\pi$ or $x = \pi - y + 2k\pi$, i.e. $x - y \in \pi\mathbb{Z}$ or $x + y \in \pi(2\mathbb{Z}+1)$.

Case $\sin x = -\sin y$: $x = -y + 2k\pi$ or $x = \pi + y + 2k\pi$, i.e. $x + y \in 2\pi\mathbb{Z}$ or $x - y \in \pi(2\mathbb{Z}+1)$.

Combining all cases, the locus is exactly $\{(x,y) : x+y \in \pi\mathbb{Z}\} \cup \{(x,y) : x - y \in \pi\mathbb{Z}\}$, a square lattice of lines of slope ± 1 at spacing π.

Question 1.140. Interviewer. Sketch $y = x \ln x$.

Sheldon. Domain $x > 0$. As $x \to 0^+$, $x \ln x \to 0$ (the polynomial dominates the logarithm). $y(1) = 0$. Differentiate: $y' = \ln x + 1$, vanishing at $x = 1/e$ with minimum $y(1/e) = -1/e$. Second derivative $y'' = 1/x > 0$, so y is convex throughout the domain. As $x \to \infty$, $y \to \infty$.

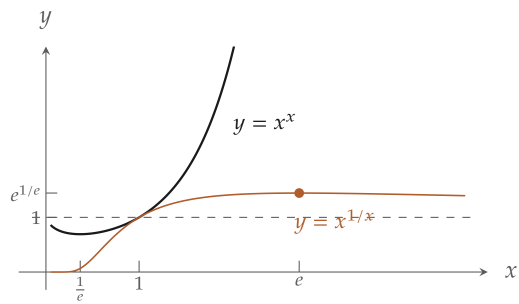


Question 1.141. Interviewer. Sketch $y = x^x$ and $y = x^{1/x}$ on the same axes.

Sheldon. Both are defined on $(0, \infty)$. For $y = x^x$ (handled in Q12): minimum $e^{-1/e} \approx 0.692$ at $x = 1/e$, value 1 at $x \to 0^+$ and $x = 1$, then super-exponential growth.

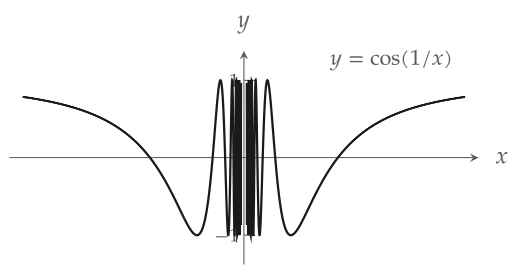
For $y = x^{1/x}$: $\ln y = \frac{\ln x}{x}$, the function from Q37. Its maximum is at $x = e$ with $y = e^{1/e} \approx 1.445$. As $x \to 0^+$, $\ln y \to -\infty$, so $y \to 0$; at $x = 1$, $y = 1$; $y \to 1^+$ as $x \to \infty$.

The two curves meet at $x = 1$ (both equal 1). For $0 < x < 1$, $x^x > x^{1/x}$ (since $x^x \to 1$ but $x^{1/x} \to 0$). For $x > 1$, x^x races past while $x^{1/x}$ peaks at e and tapers back toward 1.



Question 1.142. Interviewer. Sketch $y = \cos(1/x)$ and $y = \sin(1/x)$.

Sheldon. Both functions oscillate with bounded amplitude $|y| \leq 1$, with period (in $1/x$) of 2π, hence period in x of $2\pi x^2$ shrinking to zero as $x \to 0$. As $|x| \to 0^+$, the functions oscillate infinitely often in any neighbourhood of 0 (an essential singularity in the real-analytic sense). As $|x| \to \infty$, $1/x \to 0$, so $\cos(1/x) \to 1$ and $\sin(1/x) \to 0$. Even/odd: $\cos(1/x)$ is even, $\sin(1/x)$ is odd.



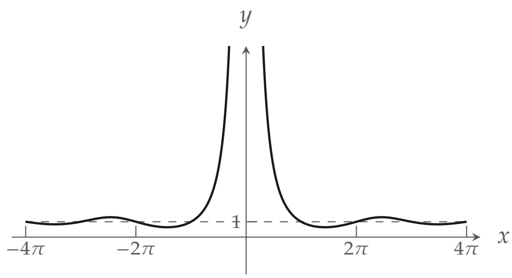
(The graph of $\sin(1/x)$ is the same shape phase-shifted by a quarter-turn in $1/x$.)

Question 1.143. Interviewer. Sketch $y = \dfrac{x + \sin x}{x - \sin x}$.

Sheldon. For large $|x|$, $\sin x$ is bounded while x grows: numerator and denominator both behave like x, so $y \to 1$. Near $x = 0$, both vanish; expand: $\sin x = x - \frac{x^3}{6} + O(x^5)$, so

$x + \sin x = 2x - \frac{x^3}{6} + O$, $x - \sin x = \frac{x^3}{6} + O$. Ratio: $y \to \frac{2x}{x^3/6} = \frac{12}{x^2}$, blowing up to $+\infty$ (even function near 0).

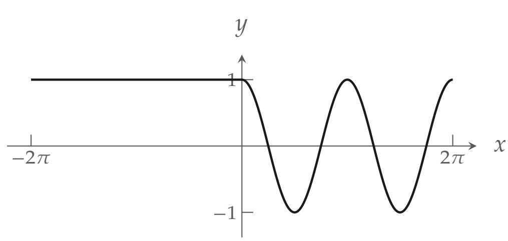
The denominator $x - \sin x$ vanishes only at $x = 0$ (since $x - \sin x$ is monotonic with single zero at the origin), so no other vertical asymptotes. The function is even (numerator and denominator each odd, ratio even).



Question 1.144. Interviewer. Sketch $y = \cos(x + |x|)$ for $-2\pi < x < 2\pi$.

Sheldon. Split by sign of x. For $x \le 0$: $|x| = -x$, so $x + |x| = 0$ and $y = \cos 0 = 1$. For $x \ge 0$: $|x| = x$, so $x + |x| = 2x$ and $y = \cos 2x$.

The graph is the constant $y = 1$ on $(-2\pi, 0]$ and the cosine curve $\cos 2x$ on $[0, 2\pi)$, joined continuously at $x = 0$ (since $\cos 0 = 1$).

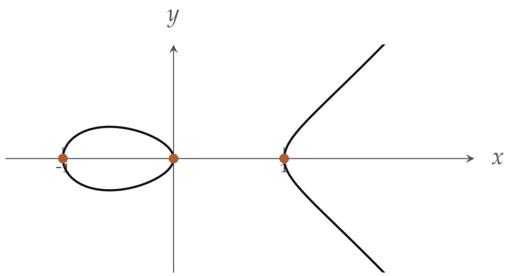


Question 1.145. Interviewer. Sketch $y = \sqrt{x^3 - x}$ and $y^2 = x^3 - x$.

SHELDON. $x^3 - x = x(x-1)(x+1) \geq 0$ on $[-1,0] \cup [1,\infty)$.

$y = \sqrt{x^3 - x}$. Single-valued and non-negative, so an *arch* on $[-1,0]$ (rising from $(-1,0)$ to a hump and back down to $(0,0)$, never dipping below the axis), plus an unbounded arm growing from $(1,0)$ to ∞. No closed loop appears here; the square root keeps only the upper half.

$y^2 = x^3 - x$. Reflect each portion across the x-axis: a closed elliptic-curve-like "loop" on $[-1,0]$ (a teardrop shape) and a pair of unbounded arms on $[1,\infty)$. This is the elliptic curve $y^2 = x(x-1)(x+1)$, much beloved of number theorists.



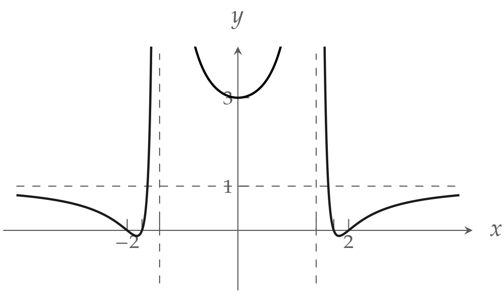
QUESTION 1.146. INTERVIEWER. Sketch $y = \dfrac{x^4 - 7x^2 + 12}{x^4 - 4x^2 + 4}$.

SHELDON. Factor numerator and denominator: $x^4 - 7x^2 + 12 = (x^2-3)(x^2-4)$ and $x^4 - 4x^2 + 4 = (x^2-2)^2$. So

$$y = \frac{(x^2-3)(x^2-4)}{(x^2-2)^2}.$$

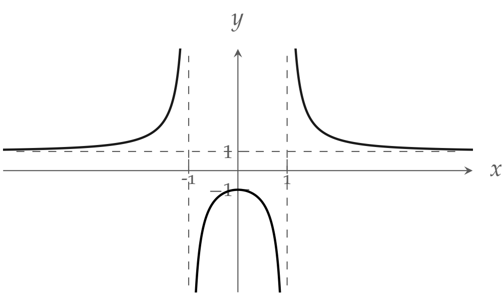
Even function. Zeros at $x = \pm\sqrt{3}$ and $x = \pm 2$. Vertical asymptotes (of even order) at $x = \pm\sqrt{2}$, where $y \to +\infty$. As $|x| \to \infty$, $y \to 1$.

Sign chart: positive far from origin, negative on $(\sqrt{3}, 2)$ and $(-2, -\sqrt{3})$, positive on $(\sqrt{2}, \sqrt{3})$, blows up at $\pm\sqrt{2}$, value at $x = 0$ is $\frac{12}{4} = 3$.



Question 1.147. Interviewer. Sketch $y = \dfrac{x^2+1}{x^2-1}$.

Sheldon. Rewrite: $y = 1 + \frac{2}{x^2-1}$. Even function. Vertical asymptotes at $x = \pm 1$. Horizontal asymptote $y = 1$. Value at $x = 0$ is -1. On $|x| > 1$ the curve lies above $y = 1$, asymptoting downwards to 1 as $|x| \to \infty$. On $|x| < 1$ the curve goes from -1 at $x = 0$ to $-\infty$ as $|x| \to 1$.

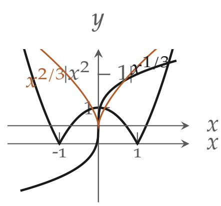


Question 1.148. Interviewer. Sketch $y = |x^2-1|$, $y = x^{1/3}$, $y = x^{2/3}$, and comment on the derivative in each case.

Sheldon. $y = |x^2-1|$. The parabola $y = x^2-1$ with its portion on $[-1, 1]$ reflected across the x-axis. Continuous everywhere, but the derivative jumps at $x = \pm 1$: it is $2x$ where $|x| > 1$ (there $|x^2-1| = x^2-1$) and $-2x$ where $|x| < 1$ (there $|x^2-1| = 1-x^2$). At $x = 1$ the slope leaps from -2 (left) to $+2$ (right), and at $x = -1$ from -2 to $+2$ likewise: a corner at each.

$y = x^{1/3}$. Defined for all real x (real cube root). Strictly increasing, odd. Derivative $\frac{1}{3}x^{-2/3} \to \infty$ as $x \to 0$: a vertical tangent at the origin.

$y = x^{2/3}$. Even function defined for all real x. Strictly increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$, with a cusp at the origin where the derivative $\frac{2}{3}x^{-1/3}$ blows up to $\pm\infty$ from each side.

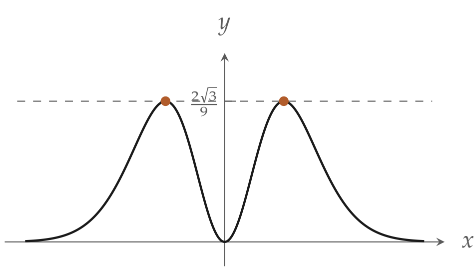


Question 1.149. Interviewer. Sketch $y = e^{-x^2} - e^{-3x^2}$.

Sheldon. Even function, $y(0) = 1 - 1 = 0$, $y \to 0$ as $|x| \to \infty$ (both exponentials vanish). Differentiate: $y' = -2xe^{-x^2} + 6xe^{-3x^2} = 2x(3e^{-3x^2} - e^{-x^2})$. Setting the bracket to zero: $3e^{-3x^2} = e^{-x^2}$, i.e. $\ln 3 = 2x^2$, $x^2 = \frac{\ln 3}{2}$, $x = \pm\sqrt{\frac{\ln 3}{2}} \approx \pm 0.741$. Maximum value at those points: substituting,

$$y_{\max} = e^{-\ln 3/2} - e^{-3\ln 3/2} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9} \approx 0.385.$$

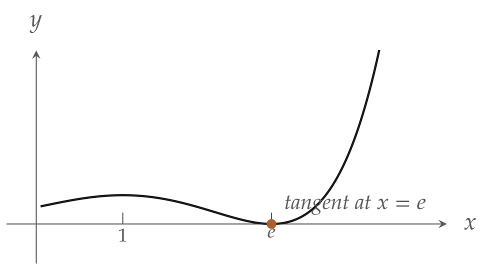
Two-bumped curve symmetric about the y-axis, vanishing at the origin and at infinity.



Question 1.150. Interviewer. By sketching, find all solutions of $x - 1 = (e - 1)\ln x$. Hence sketch $y = e^x - x^e$.

Sheldon. Both sides vanish at $x = 1$. The left side is linear, the right is logarithmic with derivative $(e-1)/x$, which equals 1 exactly at $x = e - 1 \approx 1.718$. Both curves pass through $(1, 0)$, and they also meet at $x = e$ (since $e - 1 = (e-1)\ln e$ trivially). For $x \in (1, e)$, ln rises faster initially than the line, then the line catches up: two intersections at $x = 1$ and $x = e$. By convexity arguments these are the only positive solutions.

For $y = e^x - x^e$: this rewrites as $e^x - x^e = e\ln(\text{something})$... directly, $e^x \geq x^e$ for all $x > 0$ with equality only at $x = e$ (Q42). So $y \geq 0$, with $y = 0$ uniquely at $x = e$. Tangent to the x-axis at $x = e$. Asymptotic behaviour: $y \to 1$ as $x \to 0^+$ (since $x^e \to 0$), $y \to \infty$ as $x \to \infty$.

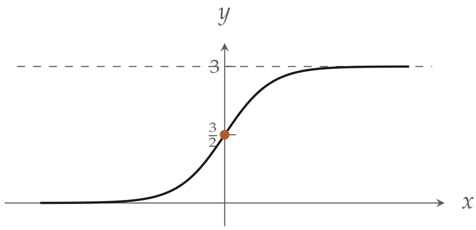


Question 1.151. Interviewer. Write $\dfrac{3e^x}{e^x + e^{-x}}$ in the form $a + \dfrac{b}{e^{2x} + 1}$, and hence sketch.

Sheldon. Multiply top and bottom by e^x:

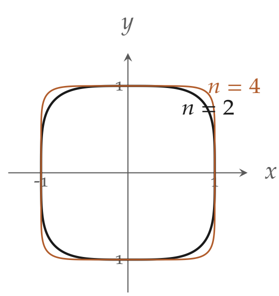
$$\frac{3e^x}{e^x + e^{-x}} = \frac{3e^{2x}}{e^{2x} + 1} = 3 \cdot \frac{e^{2x} + 1 - 1}{e^{2x} + 1} = 3 - \frac{3}{e^{2x} + 1}.$$

So $a = 3$, $b = -3$. The graph is a sigmoidal S-curve from 0 (as $x \to -\infty$) to 3 (as $x \to \infty$), passing through $(0, \frac{3}{2})$, with inflection at the origin.



Question 1.152. Interviewer. Sketch $x^{2n} + y^{2n} = 1$ for $n = 2$ and $n = 4$. Describe the limit $n \to \infty$.

Sheldon. For $n = 1$ the equation gives the unit circle. As n increases the level set bulges outward toward the corners of the unit square. The curve passes through $(\pm 1, 0)$ and $(0, \pm 1)$ regardless. For a large exponent, $|x|^{2n}$ is negligible whenever $|x| < 1$ and equals 1 only at $|x| = 1$, so along most of the curve one coordinate sits very near ± 1 (pinning the sum near 1) while the other ranges freely; only near a corner do *both* coordinates approach ± 1 together. As $n \to \infty$, $|x|^{2n} \to 0$ for $|x| < 1$ and $\to 1$ for $|x| = 1$, and the solution set converges (in Hausdorff distance) to the boundary of the unit square $\max(|x|, |y|) = 1$.

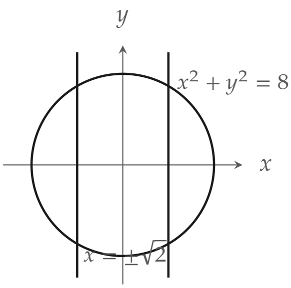


Question 1.153. Interviewer. Sketch $|3x^2 + y^2 - 12| = |x^2 - y^2 + 4|$.

Sheldon. $|A| = |B|$ iff $A = B$ or $A = -B$. The locus is the union:

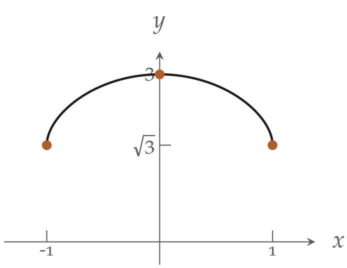
- $3x^2 + y^2 - 12 = x^2 - y^2 + 4 \implies 2x^2 + 2y^2 = 16 \implies x^2 + y^2 = 8$ (circle of radius $2\sqrt{2}$).
- $3x^2+y^2-12 = -(x^2-y^2+4) \implies 4x^2 = 8 \implies x = \pm\sqrt{2}$ (two vertical lines).

The full curve is the circle $x^2 + y^2 = 8$ together with the two vertical lines $x = \pm\sqrt{2}$.



Question 1.154. Interviewer. Sketch $y = \sqrt{1-x^2} + \sqrt{4-x^2}$.

Sheldon. Both square roots are real on $|x| \leq 1$ (the binding domain). On $[-1, 1]$ the function is the pointwise sum of the upper unit semicircle and the upper semicircle of radius 2. Even, $y(0) = 1 + 2 = 3$, $y(\pm 1) = 0 + \sqrt{3} = \sqrt{3}$. Maximum at $x = 0$.

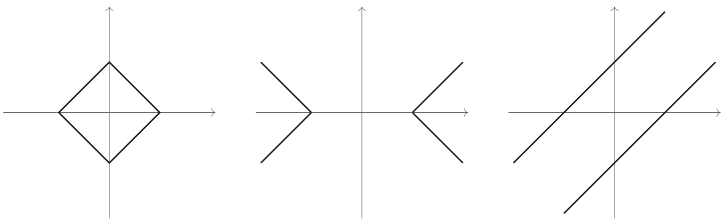


Question 1.155. Interviewer. Sketch $|x|+|y| = 1$, $|x|-|y| = 1$, $|y - x| = 1$.

Sheldon. $|x| + |y| = 1$. The unit "Manhattan" diamond, a square rotated 45° with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$.

$|x| - |y| = 1$. Resist the urge to square, which is where the careless go wrong: $|x| = 1 + |y|$ gives $x^2 = 1 + 2|y| + y^2$, not $x^2 - y^2 = 1$, so this is no hyperbola. Read it off directly. For $x \geq 0$ it says $x = 1 + |y|$, a rightward wedge with apex $(1, 0)$ and arms $y = \pm(x - 1)$, $x \geq 1$; for $x < 0$, $-x = 1 + |y|$, the mirror with apex $(-1, 0)$ and arms $y = \pm(x + 1)$, $x \leq -1$. Four straight rays, two opening right and two opening left, and not a curve in sight.

$|y - x| = 1$. Two parallel lines $y = x + 1$ and $y = x - 1$, at slant distance $1/\sqrt{2}$ from the diagonal $y = x$.



Question 1.156. Interviewer. Find the area of the region $|x| + |y| + |x + y| < 2$.

Sheldon. Split by sign of $x, y, x + y$. The expression $|x| + |y| + |x + y|$ takes simpler form per quadrant:

- First quadrant $(x, y \geq 0)$: $|x| + |y| = x + y$ and $|x + y| = x + y$, so the LHS $= 2(x + y)$. Condition: $x + y < 1$.
- Third quadrant $(x, y \leq 0)$: symmetrically, LHS $= -2(x + y)$. Condition: $x + y > -1$.
- Fourth quadrant $(x \geq 0, y \leq 0)$: split on sign of $x + y$. If $x + y \geq 0$: LHS $= x - y + (x + y) = 2x < 2$, i.e. $x < 1$. If $x + y \leq 0$: LHS $= x - y - (x + y) = -2y < 2$, i.e. $y > -1$.
- Second quadrant: by symmetry, $-x < 1$ or $y < 1$.

Combining, the region is the hexagon with vertices $(\pm 1, 0)$, $(0, \pm 1)$, $(1, -1)$, $(-1, 1)$. By the shoelace formula, the area is

$$3.$$

Question 1.157. Interviewer. Find the minimum of $|x-1| + |x-2| + |x-4| + |x-6|$.

Sheldon. The sum of distances from x to a finite set of points is minimised at any *median* of those points. With four points $\{1, 2, 4, 6\}$, every $x \in [2, 4]$ is a median, and the minimum value is the sum of distances when x lies in $[2, 4]$:

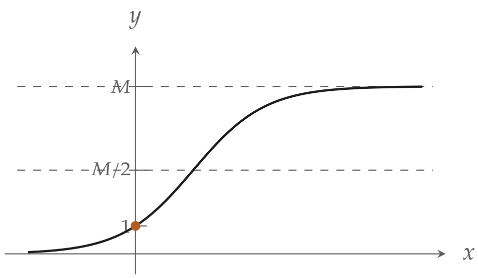
$$|x-1|+|x-2|+|x-4|+|x-6| = (x-1)+(x-2)+(4-x)+(6-x) = 7.$$

7.

Question 1.158. Interviewer. Solve $\dfrac{dy}{dx} = ky$ with $y(0) = 1$ and $k > 0$. Then sketch the solution of $\dfrac{dy}{dx} = ky(1 - \frac{y}{M})$ with the same initial condition (without solving explicitly).

Sheldon. *Linear ODE.* Separation: $\frac{dy}{y} = k\,dx$, integrating $\ln y = kx + C$, applying $y(0) = 1$ gives $y = e^{kx}$.

Logistic ODE. The right side vanishes at $y = 0$ and $y = M$ (equilibria). The shape depends on where the initial value 1 sits relative to the carrying capacity M, so assume the intended regime $M > 1$, i.e. the population starts below capacity. (If $M = 1$ the solution is the constant $y \equiv 1$; if $0 < M < 1$ the start is *above* capacity and y decreases monotonically to M with no inflection.) For $M > 1$, with $0 < y(0) = 1 < M$, y grows monotonically toward M: for $y \ll M$ the equation reduces to exponential growth $\dot{y} \approx ky$, and as y approaches M the bracket $(1 - y/M)$ shrinks and the growth slows. The inflection at $y = M/2$ (where $y'' = 0$) is traversed only when $1 < M/2$, that is $M > 2$; for $1 < M < 2$ the trajectory starts already past the steepest point and is concave throughout. Backwards in time $(x \to -\infty)$, $y \to 0^+$.



Question 1.159. Interviewer. Find $\dfrac{dy}{dx}$ when $y = \int_0^x t^8 e^t \, dt$.

Sheldon. By the Fundamental Theorem of Calculus, $\dfrac{dy}{dx} = x^8 e^x$. Integration by parts could explicitly evaluate the integral as a polynomial of degree 8 times e^x, but the question asked only for the derivative, and for the derivative the antiderivative is not the point.

Question 1.160. Interviewer. Find $f(x)$ if $\int_0^x f(t)\,dt = 3f(x) + k$.

Sheldon. Differentiate both sides with respect to x: $f(x) = 3f'(x)$, i.e. $f'(x) = \frac{1}{3}f(x)$. The general solution is $f(x) = Ae^{x/3}$.

Evaluate the original equation at $x = 0$: $0 = 3f(0) + k$, so $f(0) = -k/3$. Hence $A = -k/3$, and

$$f(x) = -\tfrac{k}{3}e^{x/3}.$$

Question 1.161. Interviewer. Find explicit expressions for $\sinh x$, $\cosh x$, and $\tanh x$.

Sheldon. By definition,

$$\sinh x = \tfrac{e^x - e^{-x}}{2}, \quad \cosh x = \tfrac{e^x + e^{-x}}{2}, \quad \tanh x = \tfrac{e^{2x}-1}{e^{2x}+1}.$$

These satisfy the Pythagorean-analogue identity $\cosh^2 x - \sinh^2 x = 1$ (expand: the cross-terms cancel, leaving $\frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{4} = \frac{4}{4} = 1$).

Question 1.162. Interviewer. Is $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ divergent? What about $1 + \frac{1}{4} + \frac{1}{9} + \cdots$? Justify by integral comparison.

Sheldon. *Harmonic divergence.* The function $1/x$ is decreasing on $[1, \infty)$, so $\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1)$. Since $\ln(n+1) \to \infty$, the harmonic series diverges.

Basel-type convergence. The function $1/x^2$ is decreasing and integrable on $[1, \infty)$: $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$. Hence $\sum_{k=1}^{n} \frac{1}{k^2} \leq 1 + \int_1^{n} \frac{1}{x^2}\,dx \leq 2$, a bounded increasing sequence which therefore converges. (Its limit is the famous $\zeta(2) = \pi^2/6$, established by Euler in 1734.)

Question 1.163. Interviewer. Shortest distance from $A(3,1)$ to $y = x^2 + 1$; from the line $y = x$ to $y = x^2 + 1$; between the curves $y = x^2 + 1$ and $x = y^2 + 1$.

Sheldon. (*a*) *Point to parabola.* Minimise $D^2 = (x-3)^2 + x^4$ over $x \in \mathbb{R}$. Differentiating, $2(x-3) + 4x^3 = 0$, i.e. $2x^3 + x - 3 = 0$; factoring, $(x-1)(2x^2 + 2x + 3) = 0$, so $x = 1$ (quadratic has negative discriminant). Closest point $(1, 2)$, distance $\sqrt{4+1} = \sqrt{5}$.

(*b*) *Line to parabola.* Point-to-line distance from $(t, t^2 + 1)$ to $x - y = 0$ is $\frac{|t-(t^2+1)|}{\sqrt{2}} = \frac{|t^2-t+1|}{\sqrt{2}}$. The expression $t^2 - t + 1$ has discriminant $-3 < 0$, so positive everywhere, with minimum value $\frac{3}{4}$ at $t = \frac{1}{2}$. Minimum distance $\frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$.

(*c*) *Curve to curve.* The second curve $x = y^2 + 1$ is the reflection of the first across $y = x$. By symmetry the minimum distance between them is twice the minimum distance from either curve to $y = x$:

$$2 \cdot \frac{3\sqrt{2}}{8} = \frac{3\sqrt{2}}{4}.$$

Question 1.164. Interviewer. Points $A(6,0)$ and $B(0,-4)$ are two vertices of a triangle. The third vertex lies on $y = x^2$. Find the coordinates minimising the triangle's area.

SHELDON. Take the third vertex $C = (t, t^2)$. The signed area is

$$\text{Area} = \tfrac{1}{2}|6(-4-t^2)+0\cdot(t^2-0)+t(0-(-4))| = \tfrac{1}{2}|6t^2-4t+24|.$$

The quadratic $6t^2-4t+24$ has discriminant $16-576 = -560 < 0$, so is strictly positive; its minimum is at $t = \frac{4}{12} = \frac{1}{3}$, value $24 - \frac{4}{3} + \frac{2}{3} = \frac{70}{3}$. Hence minimum area $\frac{35}{3}$ at $C = (\frac{1}{3}, \frac{1}{9})$.

QUESTION 1.165. INTERVIEWER. Fixed perimeter P. Max area of a triangle? Is there a right-angled triangle of smallest area with perimeter P?

SHELDON. *Max area.* By the isoperimetric inequality for triangles (equivalently, Heron's formula with Lagrange multipliers), the equilateral triangle maximises. Side $s = P/3$, area $\frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}P^2}{36}$.

Min right-angled. Let the legs be a, b and hypotenuse $c = \sqrt{a^2+b^2}$; perimeter $a+b+c = P$, area $\frac{1}{2}ab$. Fix P and let $a \to 0$: then $b \to P/2$ and area $\to 0$. The infimum is 0 but it is not attained, so *no* right-angled triangle of minimum area exists; the area can be made arbitrarily small by taking a sufficiently degenerate sliver.

QUESTION 1.166. INTERVIEWER. Prove, without directly computing, that $100 \mid 11^{10} - 1$.

SHELDON. Expand by the binomial theorem with $11 = 1 + 10$:

$$11^{10} - 1 = \sum_{k=1}^{10} \binom{10}{k} 10^k = 10 \cdot 10 + \binom{10}{2} \cdot 10^2 + \binom{10}{3} \cdot 10^3 + \cdots$$

Every summand is divisible by 100: the $k = 1$ term is $10 \cdot 10 = 100$, and for $k \geq 2$, $10^k \geq 100$. Hence $100 \mid 11^{10} - 1$.

QUESTION 1.167. INTERVIEWER. Find the sum of the coefficients of $(1 - 3x + 3x^2 - 5x^3 + 5x^4)(1 + 3x - 3x^2 + 5x^3 - 5x^4)$ after expansion.

Sheldon. The sum of the coefficients of a polynomial $P(x)$ equals $P(1)$. Evaluating,

$$P(1) = (1 - 3 + 3 - 5 + 5)(1 + 3 - 3 + 5 - 5) = 1 \cdot 1 = 1.$$

Question 1.168. Interviewer. Find a polynomial with integer coefficients whose roots include $\sqrt{2} + \sqrt{3}$.

Sheldon. Let $x = \sqrt{2} + \sqrt{3}$. Then $x^2 = 5 + 2\sqrt{6}$, so $x^2 - 5 = 2\sqrt{6}$. Squaring, $(x^2 - 5)^2 = 24$, i.e.

$$x^4 - 10x^2 + 1 = 0.$$

This is the minimal polynomial over $\mathbb{Q}$; its roots are the four Galois conjugates $\pm\sqrt{2} \pm \sqrt{3}$.

Question 1.169. Interviewer. Prove that in $(1 - x + x^2 - \cdots + x^{100})(1 + x + x^2 + \cdots + x^{100})$, no odd-degree term survives.

Sheldon. Use the closed forms:

$$1 - x + x^2 - \cdots + x^{100} = \tfrac{1+x^{101}}{1+x}, \qquad 1 + x + \cdots + x^{100} = \tfrac{1-x^{101}}{1-x}.$$

Multiplying,

$$\tfrac{(1+x^{101})(1-x^{101})}{(1+x)(1-x)} = \tfrac{1-x^{202}}{1-x^2} = \sum_{k=0}^{100} x^{2k}.$$

The product is the sum of $x^0, x^2, x^4, \ldots, x^{200}$: only even-degree terms.

Question 1.170. Interviewer. Find the integer m for which $x^4 - (3m + 2)x^2 + m^2 = 0$ has four real roots in arithmetic progression.

Sheldon. Substitute $y = x^2$: the quadratic $y^2 - (3m + 2)y + m^2 = 0$ has roots y_1, y_2 with $y_1 + y_2 = 3m + 2$, $y_1 y_2 = m^2$. The x-roots are $\pm\sqrt{y_1}, \pm\sqrt{y_2}$.

Ordered as $-\sqrt{y_2} < -\sqrt{y_1} < \sqrt{y_1} < \sqrt{y_2}$, arithmetic progression forces $\sqrt{y_2} = 3\sqrt{y_1}$, i.e. $y_2 = 9y_1$. Combining with Vieta:

$$y_1 + 9y_1 = 10y_1 = 3m + 2, \qquad 9y_1^2 = m^2 \implies 3y_1 = |m|.$$

For $m > 0$: $3y_1 = m$ and $10y_1 = 3m + 2$ give $10y_1 = 9y_1 + 2$, so $y_1 = 2$ and $m = 6$. Verify: roots $\pm\sqrt{2}, \pm 3\sqrt{2}$, common difference $2\sqrt{2}$.

For $m < 0$: $3y_1 = -m$ and $10y_1 = 3m + 2$ give $10y_1 = -9y_1 + 2$, so $y_1 = \frac{2}{19}$, non-integer $m = -\frac{6}{19}$; rejected.

Answer: $m = 6$.

Question 1.171. Interviewer. Prove the rational root theorem: if $\frac{p}{q}$ in lowest terms is a root of $a_n x^n + \cdots + a_0$, then $p \mid a_0$ and $q \mid a_n$. Hence show the n-th root of an integer is either an integer or irrational.

Sheldon. Multiply the root condition by q^n:

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

All terms except $a_0 q^n$ are divisible by p, so $p \mid a_0 q^n$; since $\gcd(p, q) = 1$, also $\gcd(p, q^n) = 1$, hence $p \mid a_0$. Symmetrically, all terms except $a_n p^n$ are divisible by q, giving $q \mid a_n$. *Application.* The n-th root of an integer k is a root of $x^n - k = 0$. By the theorem, rational roots $\frac{p}{q}$ in lowest terms must have $q \mid 1$, so $q = 1$: any rational root is an integer. Hence if $k^{1/n}$ is not an integer, it is irrational.

Question 1.172. Interviewer. If four distinct points of $y = 2x^4 + 7x^3 + 3x - 5$ are collinear, their average x-coordinate is a constant k. Find k.

Sheldon. A line $y = mx + c$ meets the curve where $2x^4 + 7x^3 + 3x - 5 = mx + c$, i.e.

$$2x^4 + 7x^3 + 0 \cdot x^2 + (3 - m)x - (5 + c) = 0.$$

By Vieta's, the sum of the four roots is $-\frac{7}{2}$ (independent of m and c). The average is therefore

$$k = \frac{-7/2}{4} = -\frac{7}{8}.$$

The constancy is the hallmark: *collinear quadruples on a degree-4 curve have a fixed average x-coordinate determined solely by the coefficients of x^4 and x^3*. This generalises to every degree-n polynomial: collinear n-tuples have x-sum $= -a_{n-1}/a_n$.

Question 1.173. Interviewer. Prove that $1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 5.

Sheldon. Modulo 5: $3 \equiv -2$ and $4 \equiv -1$. Since 99 is odd,

$$\begin{aligned} 1^{99} + 2^{99} + 3^{99} + 4^{99} + 5^{99} &\equiv 1 + 2^{99} + (-2)^{99} + (-1)^{99} + 0 \\ &\equiv 1 + 2^{99} - 2^{99} - 1 \equiv 0 \pmod 5. \end{aligned}$$

The argument generalises: for *odd* n and any $m \geq 0$, the pairing $k \leftrightarrow n-k$ gives $\sum_{k=1}^{n} k^{2m+1} \equiv 0 \pmod n$. (For even n it can fail: the middle term $(n/2)^{2m+1}$ is unpaired, and indeed $\sum_{k=1}^{2} k^3 = 9 \equiv 1 \pmod 2$.)

Question 1.174. Interviewer. Show that $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not an integer for $n > 1$.

Sheldon. Let 2^k denote the largest power of 2 not exceeding n. I claim 2^k is the *unique* multiple of 2^k in $\{1, 2, \ldots, n\}$: any other multiple would be $\geq 2 \cdot 2^k = 2^{k+1} > n$ by maximality.

Multiply H_n by $M = 2^{k-1} \cdot \operatorname{lcm}(\text{odd numbers} \leq n)$; the product $M \cdot H_n$ has integer summands for every $\frac{1}{j}$ except $j = 2^k$, for which $\frac{M}{2^k}$ has a half remaining. So $MH_n = I + \frac{1}{2}$ for some integer I: not an integer. Since M is an integer, H_n cannot be an integer either.

Question 1.175. Interviewer. For the sequence $0, 1, 1, 2, 2, 3, 3, \ldots$, find $S(n) = a_1 + \cdots + a_n$. Show $S(s+t) - S(s-t) = st$ for positive integers $s > t$.

Sheldon. The k-th term (starting from $a_1 = 0$) is $\lfloor (k-1)/2 \rfloor$. Summing,

$$S(n) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \lfloor n^2/4 \rfloor.$$

For the identity, note $s+t$ and $s-t$ have the same parity (both even if $s \equiv t \pmod 2$, both odd otherwise). In either case,

$$S(s+t) - S(s-t) = \tfrac{(s+t)^2-(s-t)^2}{4} = \tfrac{4st}{4} = st,$$

the integer residues mod 4 cancelling because the two squared values share the same remainder mod 4.

Question 1.176. Interviewer. Prove that $1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} < 3$.

Sheldon. For $k \geq 2$, $k! \geq 2^{k-1}$ (check $2! = 2$, $3! = 6 \geq 4$, and by induction $(k+1)! = (k+1) \cdot k! \geq 2 \cdot 2^{k-1} = 2^k$). Hence

$$1 + \sum_{k=2}^{n} \frac{1}{k!} \leq 1 + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = 1 + 1 = 2 < 3.$$

The actual limit, being $e - 1 \approx 1.718$, sits well below 3; the bound is rather loose. The interviewer's concession of slack is noted and charitably accepted.

Question 1.177. Interviewer. $P(x,y)$ is a symmetric polynomial with $(x-y)$ as a factor. Show that $(x-y)^2$ is also a factor.

Sheldon. Write $P(x,y) = (x-y)Q(x,y)$. Symmetry gives $P(y,x) = P(x,y)$, i.e.

$$(y-x)Q(y,x) = (x-y)Q(x,y) \implies Q(y,x) = -Q(x,y).$$

So Q is antisymmetric, hence $Q(x,x) = -Q(x,x) = 0$, so $(x-y)$ divides Q. Writing $Q(x,y) = (x-y)R(x,y)$:

$$P(x,y) = (x-y)^2 R(x,y),$$

as required.

Question 1.178. Interviewer. If $2\log(x-2y) = \log x + \log y$, find $\frac{x}{y}$.

Sheldon. The equation becomes $(x-2y)^2 = xy$, i.e. $x^2 - 5xy + 4y^2 = 0$. Factoring, $(x-y)(x-4y) = 0$, giving $x = y$ or $x = 4y$.

Check domain. All three logarithms require their arguments positive: $x > 0$, $y > 0$, $x - 2y > 0$. The case $x = y > 0$ gives $x - 2y = -y < 0$, violating the last condition; the case $x = 4y > 0$ gives $x - 2y = 2y > 0$, which is consistent.

Answer: $\frac{x}{y} = 4$.

Question 1.179. Interviewer. Let a be the repunit $11\dots1$ with m ones, and let b be 1 followed by $m-1$ zeros and a final 5. Show $ab+1$ is a perfect square and find its square root.

Sheldon. $a = \frac{10^m-1}{9}$ and $b = 10^m + 5$. Multiplying,

$$ab + 1 = \frac{(10^m-1)(10^m+5)}{9} + 1 = \frac{(10^m+2)^2}{9} = \left(\frac{10^m+2}{3}\right)^2.$$

Since $10^m + 2 \equiv 1 + 2 \equiv 0 \pmod 3$, the square root $\frac{10^m+2}{3}$ is an integer. In decimal it is $33\dots34$, namely $m-1$ threes followed by a single 4 (since $10+2 = 12$, $100+2 = 102$, $1000+2 = 1002$, divide each by 3). For $m = 2$: $a = 11$, $b = 105$, $ab + 1 = 1156 = 34^2$, as claimed.

Question 1.180. Interviewer. Prove that 10201 is composite in any base.

Sheldon. In base b, the digit string 10201 represents

$$b^4 + 2b^2 + 1 = (b^2+1)^2.$$

This is a perfect square. The string contains the digit 2, so it is a legal numeral only for base $b \geq 3$, and then $b^2 + 1 \geq 10 > 1$, making $(b^2+1)^2$ a proper composite. The trick generalises: any palindrome of the form $10\dots0d_20\dots01$ whose digits match the coefficients of a factorable polynomial is composite in every base. Here we have $(b^2+1)^2$; a curiosity mentioned by Paul Erdős in passing, though I cannot presently locate the reference, which is the kind of remark that I consider a personal failure.

Question 1.181. Interviewer. How many integers from 1 to 10^{30} are perfect squares, cubes, or fifth powers?

Sheldon. Count each set and apply inclusion–exclusion. An integer $\leq 10^{30}$ is a perfect k-th power iff it equals m^k with $m \leq 10^{30/k}$.

- Squares: 10^{15}; cubes: 10^{10}; 5th powers: 10^6.
- Square $\cap$ cube (6th power): 10^5; square $\cap$ 5th (10th power): 10^3; cube $\cap$ 5th (15th power): 10^2.
- All three (30th power): 10.

$$10^{15}+10^{10}+10^6-10^5-10^3-10^2+10 = 1\,000\,010\,000\,898\,910.$$

Question 1.182. Interviewer. If you pick 3 cards from a shuffled 52-card deck, are you more likely to see a face card than not?

Sheldon. The deck has 12 face cards and 40 non-face cards. Probability of drawing three non-face cards in a row without replacement:

$$\frac{\binom{40}{3}}{\binom{52}{3}} = \frac{40 \cdot 39 \cdot 38}{52 \cdot 51 \cdot 50} = \frac{59\,280}{132\,600} \approx 0.447.$$

So $P(\text{at least one face card}) \approx 0.553 > \frac{1}{2}$. **More likely to see a face card.**

Question 1.183. Interviewer. Evaluate $(\log_3 169)(\log_{13} 243)$ without a calculator.

SHELDON. $169 = 13^2$ and $243 = 3^5$. Apply the change-of-base logarithm rule:

$$\begin{aligned}(\log_3 13^2)(\log_{13} 3^5) &= (2\log_3 13)(5\log_{13} 3)\\ &= 10\cdot(\log_3 13)(\log_{13} 3) = 10\cdot 1 = 10,\end{aligned}$$

using $\log_a b \cdot \log_b a = 1$. A tidy illustration of the change-of-base identity.

QUESTION 1.184. INTERVIEWER. In a $4\times4\times4$ tic-tac-toe game, four symbols in a line wins. How many winning lines?

SHELDON. Standard formula for $n \times n \times n$ tic-tac-toe: the number of lines equals $\frac{(n+2)^3-n^3}{2}$. Derivation: extend each line by one cell beyond each endpoint to see it uniquely as a pair of "virtual endpoints" in $(n+2)^3 \setminus n^3$; each line has two virtual endpoints so divide by 2.

For $n = 4$:

$$\frac{6^3-4^3}{2} = \frac{216-64}{2} = 76.$$

Break-down: 48 axis-parallel lines, 24 face-diagonal lines, and 4 main-diagonal lines, summing to 76.

QUESTION 1.185. INTERVIEWER. Two positive numbers a, b with distinct first digits are multiplied. Can the first digit of ab lie strictly between the first digits of a and b?

SHELDON. Write $a = A\cdot 10^p$, $b = B\cdot 10^q$ with $A, B \in [1, 10)$ the significands. Then $AB \in [d_a d_b, (d_a+1)(d_b+1))$, where $d_a = \lfloor A \rfloor$ and $d_b = \lfloor B \rfloor$ are the first digits. The first digit of ab is $\lfloor AB \rfloor$ if $AB < 10$, else $\lfloor AB/10 \rfloor$.

Assume WLOG $d_a < d_b$. Suppose the first digit of ab lies in $\{d_a+1, \dots, d_b-1\}$. Two cases:

- $AB < 10$: then $\lfloor AB \rfloor < d_b$ would require $AB < d_b$, but $AB \geq d_a d_b \geq d_b$ (since $d_a \geq 1$). Contradiction.

- $AB \geq 10$: then $\lfloor AB/10 \rfloor > d_a$ requires $AB \geq 10(d_a + 1)$, but $AB < (d_a+1)(d_b+1) \leq 10(d_a+1)$ (since $d_b+1 \leq 10$). Contradiction.

In both cases the claim fails: the first digit of ab is either $\leq d_a$ or $\geq d_b$, never strictly between.

Question 1.186. Interviewer. A gambler bets half his money on each coin toss; he wins on heads, loses on tails. After many games, with equal wins and losses, is he ahead, behind, or even?

Sheldon. Each win multiplies the gambler's money by $1 + \frac{1}{2} = \frac{3}{2}$; each loss by $1 - \frac{1}{2} = \frac{1}{2}$. After n wins and n losses (in any order, since the operations commute),

$$M_{\text{final}} = M_0 \cdot \left(\tfrac{3}{2}\right)^n \left(\tfrac{1}{2}\right)^n = M_0 \cdot \left(\tfrac{3}{4}\right)^n \to 0.$$

He loses money, and a lot of it. This is the classic asymmetric geometric-mean trap: arithmetic gain per win does not equal arithmetic loss per loss, and the geometric mean determines long-run wealth.

Question 1.187. Interviewer. Alice and Bob play a fair £1 game. Alice starts with £a, Bob with £b. What is the probability Alice wins all Bob's money?

Sheldon. Classical gambler's ruin. Let p_k denote Alice's probability of winning starting from fortune k (with total $T = a+b$). She plays until $k = 0$ (ruin) or $k = T$ (victory). The fair game gives $p_k = \frac{1}{2}(p_{k-1} + p_{k+1})$: p_k is linear in k. With boundary $p_0 = 0$ and $p_T = 1$,

$$p_k = \frac{k}{T}, \qquad \text{Alice's winning probability} = p_a = \frac{a}{a+b}.$$

In the unfair case (win probability $p \neq \frac{1}{2}$) the answer becomes $\frac{1-(q/p)^a}{1-(q/p)^{a+b}}$ with $q = 1 - p$; I note this for completeness, though the interviewer did not ask.

Question 1.188. Interviewer. Find $F(x)$ satisfying $x^2F(x) + F(1-x) = 2x - x^4$.

Sheldon. Try $F(x) = 1 - x^2$; then $F(1-x) = 1-(1-x)^2 = 2x - x^2$. Substituting,

$$x^2(1-x^2) + (2x - x^2) = x^2 - x^4 + 2x - x^2 = 2x - x^4.$$

Answer: $F(x) = 1 - x^2$.

(Systematically: replace x by $1-x$ in the original to obtain a second equation, then solve the linear system in $F(x), F(1-x)$. The ansatz short-cuts this.)

Question 1.189. Interviewer. How many permutations $(s_1, \dots, s_n)$ of $(1, \dots, n)$ satisfy $s_k > k - 3$ for all k?

Sheldon. The constraint forces $s_k \in \{k-2, k-1, k, \dots, n\}$. Equivalently, element j may be placed only in positions $\{1, 2, \dots, \min(j+2, n)\}$.

Place elements $1, 2, \dots, n$ in sequence. When placing element j, the allowed set has $\min(j+2, n)$ positions; of these, $j-1$ are already occupied. Available choices:

- For $j \le n-2$: $(j+2)-(j-1) = 3$ choices.
- For $j = n-1$: $n-(n-2) = 2$ choices.
- For $j = n$: 1 choice.

Multiplying, the count is

$$\underbrace{3 \cdot 3 \cdots 3}_{n-2 \text{ factors}} \cdot 2 \cdot 1 = 2 \cdot 3^{n-2} \quad (n \ge 2).$$

Question 1.190. Interviewer. Is $\dfrac{1}{2n+1}\binom{2n}{n}$ always integer-valued for positive integer n?

Sheldon. No. At $n = 1$: $\frac{1}{3}\binom{2}{1} = \frac{2}{3}$, not an integer. The expression resembles the Catalan number but is missing a factor:

Catalan $C_n = \frac{1}{n+1}\binom{2n}{n}$ (divisor $n+1$) is always integer, and the equivalent form $\frac{1}{2n+1}\binom{2n+1}{n}$ also works. The divisor $2n+1$ on $\binom{2n}{n}$, as posed, gives integers only for specific n and not in general.

Question 1.191. Interviewer. A ladder is carried around a right-angle turn between corridors of widths m and n. What is the maximum ladder length?

Sheldon. Parametrise the ladder by the angle θ it makes with one wall. A ladder of length L fits at angle θ iff $L \le \frac{m}{\sin\theta} + \frac{n}{\cos\theta}$; the maximum L that fits at *every* angle is the minimum of this expression over $\theta \in (0, \frac{\pi}{2})$.

Differentiating and setting to zero: $\tan^3\theta = \frac{m}{n}$, i.e. $\tan\theta = (m/n)^{1/3}$. Substituting back,

$$L_{\max} = \left(m^{2/3} + n^{2/3}\right)^{3/2}.$$

Classical "moving sofa"–adjacent problem; the full Gerver sofa constant is still (as of my last update) an active open question.

Question 1.192. Interviewer. Count bracelets of 11 beads in 3 colours with rotations identified but reflections not.

Sheldon. With reflections *not* identified, this is the *necklace* count (despite the nominal "bracelet"). By Burnside's lemma over the cyclic group of order 11 (prime),

$$\frac{1}{11}\sum_{d|11}\varphi(d)\cdot 3^{11/d} = \frac{3^{11} + 10\cdot 3^1}{11} = \frac{177\,147 + 30}{11} = 16\,107.$$

The divisors of 11 being $\{1, 11\}$ makes the sum particularly clean.

Question 1.193. Interviewer. (a) A fair six-sided die is rolled 12 times; probability of getting each face exactly twice?

(b) A fair ten-sided die rolled 4 times; probability the sequence is strictly increasing?

SHELDON. (*a*) Multinomial: the favourable count is $\frac{12!}{(2!)^6}$, and the total is 6^{12}:

$$P = \frac{12!}{(2!)^6 \cdot 6^{12}} = \frac{479\,001\,600}{64 \cdot 2\,176\,782\,336} \approx 0.00344.$$

(*b*) A strictly increasing 4-roll sequence corresponds to choosing 4 distinct values from $\{1, \ldots, 10\}$ (then ordering them ascending): $\binom{10}{4}$ favourable, 10^4 total:

$$P = \frac{\binom{10}{4}}{10^4} = \frac{210}{10\,000} = 0.021.$$

QUESTION 1.194. INTERVIEWER. Sketch $x = ay^2$ for $a \geq 0$. If a circle meets this parabola in four points y_1, y_2, y_3, y_4, prove $y_1 + y_2 + y_3 + y_4 = 0$.

SHELDON. The parabola opens to the right; for $a = 0$ it degenerates to the y-axis. Substituting $x = ay^2$ into the circle $(x-h)^2 + (y-k)^2 = R^2$:

$$(ay^2 - h)^2 + (y-k)^2 = R^2 \iff a^2y^4 + (1-2ah)y^2 - 2ky + C = 0,$$

where $C = h^2 + k^2 - R^2$. This is a quartic in y with *no* y^3 *term*. By Vieta's, the sum of the four roots is $-(\text{coeff of } y^3)/(\text{coeff of } y^4) = 0$.

QUESTION 1.195. INTERVIEWER. What are the remainders of 4^n modulo 3?

SHELDON. Since $4 \equiv 1 \pmod 3$, $4^n \equiv 1^n \equiv 1 \pmod 3$ for every $n \geq 0$. The remainder is always 1.

QUESTION 1.196. INTERVIEWER. From the 4×4 grid with entries $1, 2, \ldots, 16$ arranged in rows (entry $(i, j) = 4(i-1) + j$), place four counters so no two share a row or column. Why does the sum always equal 34?

SHELDON. Choose a permutation σ of $\{1,2,3,4\}$ and place counters at positions $(i,\sigma(i))$. The sum of the four entries is

$$\sum_{i=1}^{4}[4(i-1)+\sigma(i)] = 4(0+1+2+3)+(1+2+3+4) = 24+10 = 34.$$

The row part contributes $4 \cdot \binom{4}{2} = 24$ regardless of σ, and the column part contributes $1+2+3+4 = 10$ because σ is a permutation. The sum is σ-independent: any such placement gives 34.

QUESTION 1.197. INTERVIEWER. Find all real solutions of $\log_{2x}(8x^2+8x) = 3$.

SHELDON. Exponentiate: $(2x)^3 = 8x^2 + 8x$, i.e. $8x^3 = 8x^2 + 8x$, so $x^3 - x^2 - x = 0$, i.e. $x(x^2 - x - 1) = 0$.

Discarding $x = 0$ (invalid base), the quadratic gives $x = \frac{1 \pm \sqrt{5}}{2}$. The negative root $\frac{1-\sqrt{5}}{2} \approx -0.618$ gives base $2x < 0$, also invalid.

Solution: $x = \frac{1+\sqrt{5}}{2} = \varphi$, the golden ratio.

QUESTION 1.198. INTERVIEWER. A pencil on a rope of length r is attached to a pivot that moves along the perimeter of a square frame of side $2a$. What is the area of the shaded region swept by the pencil?

SHELDON. The swept region is the Minkowski sum of the square's *boundary* with a disc of radius r: all points within r of the boundary.

Case $r \geq a$. The entire square plus the exterior dilation is covered. Area = Minkowski sum with closed disc = square with corners rounded to radius r:

$$4a^2 + 8ar + \pi r^2.$$

Case $r < a$. The inner square of side $2(a-r)$ is not reached (points more than r from every edge). Subtract its area from the previous:

$$(4a^2 + 8ar + \pi r^2) - 4(a-r)^2 = 16ar + (\pi - 4)r^2.$$

Verify continuity at $r = a$: both expressions give $12a^2 + \pi a^2$, as required.

Question 1.199. Interviewer. On a torus-shaped space station with no gravity, spaceman A wants to throw a spanner to spaceman B, diametrically opposite. What angle and speed should A choose?

Sheldon. Assume: zero gravity, no air resistance, negligible rotation of the station, no other forces. In the inertial frame of the spacemen, the spanner travels in a straight line. Aim directly at B (through the central hole of the torus, along the chord connecting the two antipodal points). Any positive speed will deliver the spanner; the choice affects only time-of-flight.

If one wishes to be pedantic (and I do): the line of sight from A to B might be blocked by the torus itself depending on B's exact angular position around the minor circle. In that case, throw along a tangent path that wraps, or choose a speed sufficient for the curved geodesic; absent further constraints, I decline to over-engineer the answer.

Question 1.200. Interviewer. A telephone cable encircles the Earth along its equator. How much additional cable is needed to raise it uniformly to the top of 10-metre poles?

Sheldon. The Earth's radius is approximately $R = 6.37 \times 10^6$ metres, but it does not matter. The original cable length is $2\pi R$; the raised cable has length $2\pi(R + 10)$. The difference is

$$2\pi(R + 10) - 2\pi R = 20\pi \approx 62.8 \text{ metres.}$$

Independent of the planet. The same calculation applies whether the sphere is the Earth or a marble: raising a great-circle cable by one metre always requires precisely 2π additional metres. Counterintuitive until one reads the formula, at which point it becomes tautological. My preferred kind of answer.

Question 1.201. Interviewer. A thin hoop of diameter d is thrown onto an infinite chessboard with squares of side L. What is the probability that the hoop encloses two colours?

Sheldon. Read "encloses two colours" as: the disc bounded by the hoop, of radius $r = d/2$, covers parts of squares of both colours (not merely that the thin rim crosses a grid line). That happens iff the disc is not contained in a single square. By translational symmetry, condition on the disc's centre relative to its containing square. The centre yields a single-coloured disc iff it lies in the inner square of side $L-d$ (centre at distance $\geq r$ from every edge); this requires $d < L$.

$$P(\text{two colours}) = \begin{cases} 1 - (1 - d/L)^2 = \dfrac{2d}{L} - \dfrac{d^2}{L^2}, & d < L, \\ 1, & d \geq L. \end{cases}$$

Question 1.202. Interviewer. An infinite floor is tiled with regular hexagons of side L, no two same-coloured tiles touching. A hoop of diameter d is thrown. Probability the hoop encloses more than one colour?

Sheldon. The inradius (apothem) of a regular hexagon of side L is $L\sqrt{3}/2$. The disc enclosed by the hoop is single-coloured iff its centre lies in the smaller hexagon obtained by reducing the apothem by $r = d/2$, i.e. the hexagon of inradius $L\sqrt{3}/2 - d/2$, equivalently of side $L - d/\sqrt{3}$.

The area of a regular hexagon scales with the square of its side, so

$$P(\text{more than one colour}) = 1 - \left(1 - \frac{d}{L\sqrt{3}}\right)^2 \quad \text{for } d < L\sqrt{3},$$

and equals 1 when $d \geq L\sqrt{3}$.

Question 1.203. Interviewer. Find the volume of the largest cube that fits entirely inside a sphere of unit volume.

Sheldon. A cube inscribed in a sphere has its space diagonal equal to the sphere's diameter: side s, diagonal $s\sqrt{3}$, so $s = 2R/\sqrt{3}$ for sphere of radius R. With sphere volume $\frac{4}{3}\pi R^3 = 1$, $R^3 = \frac{3}{4\pi}$, and

$$V_{\text{cube}} = s^3 = \left(\frac{2R}{\sqrt{3}}\right)^3 = \frac{8R^3}{3\sqrt{3}} = \frac{8}{3\sqrt{3}} \cdot \frac{3}{4\pi} = \frac{2}{\pi\sqrt{3}} = \frac{2\sqrt{3}}{3\pi} \approx 0.367.$$

Question 1.204. Interviewer. Area of a regular n-gon inscribed in a circle of radius r?

Sheldon. n congruent isosceles triangles meet at the centre, each with apex angle $2\pi/n$ and equal sides of length r. Total area

$$A_n = \frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right) \xrightarrow{n\to\infty} \pi r^2.$$

The same formula appeared in Q4; the only novelty is the recurrence in interview frequency.

Question 1.205. Interviewer. For a circle inscribed in a regular n-gon, what is the smallest n such that the area outside the circle but inside the polygon is at most $\frac{1}{1000}$ of the circle's area?

Sheldon. Inscribed circle of inradius r in a regular n-gon: the polygon has area $nr^2\tan(\pi/n)$, the circle has area πr^2. The condition

$$\frac{nr^2\tan(\pi/n) - \pi r^2}{\pi r^2} \leq \frac{1}{1000} \iff \frac{n\tan(\pi/n)}{\pi} \leq \frac{1001}{1000}.$$

Taylor expansion: $n\tan(\pi/n) \approx \pi + \frac{\pi^3}{3n^2}$. So the LHS $\approx 1 + \frac{\pi^2}{3n^2}$, requiring $n^2 \geq 1000\pi^2/3 \approx 3290$, i.e. $n \geq 58$.

Verify numerically: at $n = 57$, $n\tan(\pi/n)/\pi \approx 1.00102 > \frac{1001}{1000}$. At $n = 58$, $\approx 1.00077 \leq \frac{1001}{1000}$. So the answer is $n = 58$.

Question 1.206. Interviewer. Give a vector proof that a triangle inscribed in a semicircle has its included angle equal to $\pi/2$.

Sheldon. Place the circle's centre at the origin and the diameter along the x-axis: vertices $A = -\mathbf{e}$, $B = \mathbf{e}$ for unit vector $\mathbf{e}$. Let C be any point on the semicircle, so $|\mathbf{C}| = 1 = |\mathbf{e}|$. The vectors from C are $\overrightarrow{CA} = -\mathbf{e} - \mathbf{C}$ and $\overrightarrow{CB} = \mathbf{e} - \mathbf{C}$. Their dot product:

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = -\mathbf{e} \cdot \mathbf{e} + \mathbf{C} \cdot \mathbf{C} = -1 + 1 = 0.$$

Hence the angle at C is $\pi/2$. Thales of Miletus, c. 600 BCE.

Question 1.207. Interviewer. Tile an 8×8 chessboard with two corner squares removed by dominoes. Possible if (a) the corners are adjacent, or (b) diagonally opposite?

Sheldon. Each domino covers exactly one black and one white square.

(a) *Adjacent corners removed.* Adjacent corners differ in colour, so 31 black and 31 white squares remain. The colour count is balanced, and an explicit tiling is easy to exhibit (the standard "brick" pattern with adjustments at the missing corners). Possible.

(b) *Diagonally opposite corners removed.* Opposite corners share colour, so 32 of one colour and 30 of the other remain. Each domino covers exactly one of each, so any tiling would need equal counts. Impossible.

Question 1.208. Interviewer. Sketch $y^2 = \sin x$.

Sheldon. Defined where $\sin x \geq 0$, i.e. on $\bigcup_{k\in\mathbb{Z}}[2k\pi, (2k+1)\pi]$. On each such interval the curve is a teardrop (closed loop pinched at the endpoints): $y = \pm\sqrt{\sin x}$, both branches meeting the axis at $x = k\pi$ and reaching $y = \pm 1$ at the midpoints. Between consecutive "humps" (i.e., on $[(2k+1)\pi, (2k+2)\pi]$) the curve has no points: $\sin x < 0$ there.

Question 1.209. Interviewer. An equilateral triangle has side 1. Find the area of the inscribed square (with one side on the triangle's base and the other two vertices on the slanting sides).

Sheldon. Place the triangle with vertices $(0,0)$, $(1,0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let the inscribed square have side s, with bottom-left corner at $(a, 0)$. The top-left corner (a, s) lies on the left slope $y = x\sqrt{3}$, giving $s = a\sqrt{3}$, i.e. $a = s/\sqrt{3}$. The top-right corner $(a+s, s)$ lies on the right slope $y = (1-x)\sqrt{3}$, giving $s = (1-a-s)\sqrt{3}$, i.e. $1 - s/\sqrt{3} - s = s/\sqrt{3}$, so $s(1 + 2/\sqrt{3}) = 1$, hence

$$s = \frac{\sqrt{3}}{2+\sqrt{3}} = 2\sqrt{3} - 3.$$

Area $= s^2 = (2\sqrt{3}-3)^2 = 21 - 12\sqrt{3} \approx 0.215$.

Question 1.210. Interviewer. Two bags hold m and n balls. Operations: (a) remove an equal number from each; (b) double one bag. Can the bags always be emptied? What if (b) is replaced by tripling, (b′)?

Sheldon. First, a dead end to respect: once one bag is empty and the other is not, the game is lost, for subtraction needs both bags and doubling (or tripling) leaves 0 at 0. So "empty the bags" means reach a square state (k, k) and subtract once. Assume, then, both bags start non-empty.

Doubling. Always possible. Take $1 \le m \le n$. Double the smaller bag until it lands in $(\frac{n}{2}, n]$: doubling eventually passes $\frac{n}{2}$, and the first time it does, $m \cdot 2^t \le 2 \cdot \frac{n}{2} = n$, so it lands in the interval. If $m \cdot 2^t = n$ the bags are equal; subtract and finish. Otherwise the pair is (s, n) with $\frac{n}{2} < s < n$; subtract $s - 1$ from both, leaving $(1, n - s + 1)$ with $n - s + 1 < \frac{n}{2} + 1 \le n$. The larger bag has strictly shrunk and both stay positive, so by descent we reach $(1, 1)$ and then $(0, 0)$.

Tripling. Here $m + n \bmod 2$ is invariant: tripling a bag changes the total by an even amount ($2m$ or $2n$), and subtracting k from both changes it by $-2k$. The target $(0, 0)$ is even, so the bags can be emptied *only if* $m+n$ is even. The condition is also sufficient. With $1 \le m \le n$ and $n-m$ even: if the smaller bag is at least half the larger, first subtract enough to bring it strictly under half (legal because $n - m \ge 2$); then triple the smaller bag, which moves it toward the larger and so strictly shrinks the even gap $n - m$ while keeping both positive. Repeating drives the gap to 0, reaching a square state. For instance $(1, 5) \to (3, 5) \to (1, 3) \to (3, 3) \to (0, 0)$.

Question 1.211. Interviewer. Prove there are infinitely many primitive Pythagorean triples.

Sheldon. The family $(2k + 1,\ 2k^2 + 2k,\ 2k^2 + 2k + 1)$ for $k = 1, 2, 3, \ldots$ gives a primitive triple for each k: the odd leg is $2k + 1$ and the hypotenuse exceeds the even leg by 1, so $\gcd(\text{hypotenuse}, \text{even leg}) = 1$, and any common factor of the odd leg and an even quantity must divide 1. Verification: $(2k + 1)^2 + (2k^2 + 2k)^2 = 4k^2 + 4k + 1 + 4k^4 + 8k^3 + 4k^2 = (2k^2 + 2k + 1)^2$.

The sequence $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(9, 40, 41)$, …is therefore an infinite list of pairwise distinct primitive triples.

Question 1.212. Interviewer. Straight lines are drawn in the plane. Show that the resulting regions can be 2-coloured so adjacent regions differ.

Sheldon. Induction on the number of lines.

Base. Zero lines: one region, colour it red.

Step. Assume an arrangement of k lines has a valid 2-colouring. Add a $(k+1)$-st line ℓ. Flip the colour of every region on one side of ℓ. Adjacent regions:

- Both on the same side of ℓ: the line separating them is one of the original k, and both colours flipped, so they remain differently coloured.
- On opposite sides of ℓ: the line separating them *is* ℓ. They were the same region before ℓ was added (so identically coloured); now one is flipped, so they differ.

The 2-colouring extends.

Question 1.213. Interviewer. For a 4-digit number n (not all four digits equal), let $K(n)$ be the difference between the largest and smallest rearrangements of its digits (with leading zeros allowed). Show that iterating K reaches 6174 within 7 steps.

Sheldon. This is *Kaprekar's routine* (1949). The Kaprekar map sends the set of 4-digit "patterns" (multisets of four digits, not all equal) to itself, and 6174 is the unique fixed point: $9741 - 1479 = 6174$, and the digits of 6174 rearranged give back $7641 - 1467 = 6174$.

The number of distinct 4-digit patterns (multisets of four digits, not all equal) is $\binom{13}{4} - 10 = 705$, but the map collapses the orbit space at once: a rearrangement preserves the digit sum, so $K(n)$ is always a multiple of 9, and one checks that every output is one of only 30 reachable patterns. The claim is then a finite check, which I have run: *every* one of the 705 patterns reaches 6174 in at most 7 steps, and 7 is attained. A worst case is

$$1004 \to 4086 \to 8172 \to 7443 \to 3996 \to 6264 \to 4176 \to 6174,$$

seven applications of K. (A common textbook chain, $9998 \to 8991 \to \cdots$, is misquoted: $K(9998) = 9{,}981 - 1{,}899$ is wrong because 9998 sorts to 9,998 and 8,999, giving $K(9998) = 0999$; the true chain $9998 \to 0999 \to 8991 \to 8082 \to 8532 \to 6174$ takes only five steps.) The fixed point is genuine: $K(6174) = 7{,}641 - 1{,}467 = 6174$.

The proof is computational rather than structural. I personally find non-structural proofs unsatisfying, but Kaprekar lacked the computer that would have made this routine; given the constraint, his observation is impressive.

Question 1.214. Interviewer. Gnomes have friendships; friendship is symmetric. Show the number of gnomes with an odd number of friends is even.

Sheldon. The handshake lemma. Sum the degrees over all gnomes: each friendship contributes 2 to the total (once at each endpoint), so the sum is even. The sum of even-degree gnomes' degrees is even, so the sum of odd-degree gnomes' degrees is also even. A sum of odd numbers is even iff the count of summands is even. Therefore the number of odd-degree gnomes is even. A foundational result in graph theory; the term "handshake lemma" is, I admit, charmingly informal for what is essentially conservation of edge-endpoints.

Question 1.215. Interviewer. Prove that for coprime integers A, B, there exist integers s, t with $As + Bt = 1$.

Sheldon. Consider the set $S = \{As + Bt : s, t \in \mathbb{Z}\}$. S is closed under addition and additive inverses, contains A and B, and contains 0. Let d be the smallest positive element of S (well-defined since S contains $|A|$ and $|B|$, positive integers).

By division of A by d: $A = qd + r$ with $0 \le r < d$. Then $r = A - qd \in S$ (since $A, d \in S$ and S is closed under integer combinations). The minimality of d forces $r = 0$, so $d \mid A$. By symmetry $d \mid B$. Hence $d \mid \gcd(A, B) = 1$, so $d = 1$. Since $1 \in S$, there exist s, t with $As + Bt = 1$.

Question 1.216. Interviewer. Evaluate $\int_0^1 \frac{1}{x + \sqrt{1+x^2}}\,dx$.

Sheldon. Rationalise by multiplying numerator and denominator by $\sqrt{1+x^2} - x$:

$$\frac{1}{x+\sqrt{1+x^2}} = \frac{\sqrt{1+x^2}-x}{(1+x^2)-x^2} = \sqrt{1+x^2} - x.$$

Hence

$$\int_0^1 \left(\sqrt{1+x^2} - x\right) dx = \left[\tfrac{x\sqrt{1+x^2}+\sinh^{-1} x}{2} - \tfrac{x^2}{2}\right]_0^1$$
$$= \tfrac{\sqrt{2}-1+\ln(1+\sqrt{2})}{2}.$$

Question 1.217. Interviewer. Evaluate $\lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right)$.

Sheldon. Recognise as a Riemann sum: rewrite each summand $\frac{1}{n+k} = \frac{1}{n} \cdot \frac{1}{1+k/n}$ for $k = 1, \ldots, n$. The sum is

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+k/n} \xrightarrow{n\to\infty} \int_0^1 \frac{dx}{1+x} = \ln 2.$$

Question 1.218. Interviewer. Find $\int_1^n \frac{(-1)^{\lfloor x \rfloor}}{\lfloor x \rfloor}\,dx$ for $n \in \mathbb{N}$.

Sheldon. On each unit interval $[k, k+1)$ for $k \in \{1, 2, \ldots, n-1\}$, the floor is constant and the integrand is the constant $(-1)^k/k$.

Integrating over each yields $(-1)^k/k$, and summing,

$$\int_1^n \frac{(-1)^{\lfloor x \rfloor}}{\lfloor x \rfloor}\,dx = \sum_{k=1}^{n-1} \frac{(-1)^k}{k} = -\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \cdots + \tfrac{(-1)^{n-2}}{n-1}\right).$$

As $n \to \infty$ the value approaches $-\ln 2$, the alternating harmonic sum.

Question 1.219. Interviewer. Prove that every prime has infinitely many multiples in the Fibonacci sequence, and at constant frequency.

Sheldon. The Fibonacci sequence modulo p is periodic: only finitely many residue pairs (F_n, F_{n+1}) mod p exist and the recurrence runs both ways, so the residues cycle with some period $\pi(p)$, the *Pisano period*. As $F_0 = 0$, the residue 0 recurs, so $p \mid F_n$ for infinitely many n.

The frequency, however, is set not by $\pi(p)$ but by the *rank of apparition* $z(p)$, the least $n > 0$ with $p \mid F_n$. From the identity $\gcd(F_a, F_b) = F_{\gcd(a,b)}$ one gets $p \mid F_n \iff z(p) \mid n$: the indices of Fibonacci multiples of p are exactly the multiples of $z(p)$. They form a single arithmetic progression of common difference $z(p)$ and constant density $1/z(p)$. (Always $z(p) \mid \pi(p)$, with quotient 1, 2, or 4; counting by the whole period would overstate the gap.)

The primes do not merely recur, they proliferate: Carmichael's theorem (1913) says every F_n with $n \notin \{1, 2, 6, 12\}$ carries a *primitive* prime factor dividing no earlier Fibonacci. So infinitely many distinct primes appear, each making its debut at its own rank of apparition.

Question 1.220. Interviewer. Find all integer solutions of $4x^2 = y^3 + 1$.

Sheldon. Rearrange: $y^3 = 4x^2 - 1 = (2x-1)(2x+1)$. The two factors $2x - 1$ and $2x + 1$ are coprime (they differ by 2 and both are odd). Their product is a perfect cube, so each factor is itself a perfect cube up to sign:

$$2x - 1 = a^3, \qquad 2x + 1 = b^3, \qquad b^3 - a^3 = 2.$$

Factor: $(b - a)(a^2 + ab + b^2) = 2$. Integer factorisations: $(b - a, a^2 + ab + b^2) \in \{(1, 2), (2, 1), (-1, -2), (-2, -1)\}$. The form $a^2 + ab + b^2$ is non-negative for all real a, b, so the negative options are impossible.

Case $(1, 2)$*:* $b = a + 1$ and $3a^2 + 3a + 1 = 2$, i.e. $3a^2 + 3a - 1 = 0$, no integer solution.

Case $(2, 1)$*:* $b = a + 2$ and $3a^2 + 6a + 4 = 1$, i.e. $(a + 1)^2 = 0$, so $a = -1, b = 1$. Then $2x - 1 = -1, 2x + 1 = 1$, giving $x = 0$ and $y = ab = -1$. Verification: $4 \cdot 0 = 0 = (-1)^3 + 1$.

The unique integer solution is $(x, y) = (0, -1)$.

Question 1.221. Interviewer. $A = \{0, 1, \ldots, 2^n\}$, $B = \{0, 1, \ldots, n\}$. How many weakly increasing $g : A \to B$ satisfy $g(2^k) = k$ for all $k \in B$?

Sheldon. The powers-of-two constraint pins g at $n + 1$ points: $g(1) = 0, g(2) = 1, \ldots, g(2^n) = n$. The value $g(0)$ is forced by monotonicity: $0 \leq 0 = g(1)$ allows $g(0) = 0$ only.

Between consecutive powers 2^k and 2^{k+1}, g increases (weakly) from k to $k + 1$ across the $2^k + 1$ integers $\{2^k, 2^k + 1, \ldots, 2^{k+1}\}$, taking values in $\{k, k + 1\}$. The position at which g switches from k to $k + 1$ can be any of 2^k choices.

Multiplying over $k = 0, 1, \ldots, n - 1$:

$$\prod_{k=0}^{n-1} 2^k = 2^{0+1+\cdots+(n-1)} = 2^{n(n-1)/2}.$$

Question 1.222. Interviewer. Find all $f : \mathbb{Z} \to \mathbb{Z}$ with $f(a) + f(b) = f(f(a + b))$.

Sheldon. Let $c = f(0)$. Set $b = 0$: $f(a) + c = f(f(a))$. Set $a = 0$: identical relation. So $f \circ f = f + c$.

Now rewrite the original as $f(a) + f(b) = f(a + b) + c$ (using $f \circ f = f + c$). Let $g(a) = f(a) - c$. Then g is additive: $g(a+b) = g(a) + g(b)$. Over $\mathbb{Z}$, additive functions are linear: $g(a) = ka$ for $k = g(1)$.

So $f(a) = ka + c$. Plugging back into $f \circ f = f + c$: $f(f(a)) = k(ka + c) + c = k^2a + kc + c$, must equal $f(a) + c = ka + 2c$. Matching: $k^2 = k$ so $k \in \{0, 1\}$; and $kc + c = 2c$ so $(k - 1)c = 0$. Either $k = 1$ (any c), or $k = 0$ and $c = 0$ (so $f \equiv 0$).

All solutions: $f(a) = a+c$ for any $c \in \mathbb{Z}$, or $f \equiv 0$. The variants (b) and (c), with $f(2b)$ or $f(2a)$ in place of $f(b)$, further force $f(2x) = f(x)$, collapsing the translation family; (b) admits only $f \equiv 0$, while (c) admits $f \equiv 0$ and $f(x) = 2x$.

Question 1.223. Interviewer. Given n points in the plane, show that one can select at least $\sqrt{n}$ of them forming no equilateral triangle.

Sheldon. Each pair of points determines at most 2 potential third vertices forming an equilateral triangle, so the number of such triangles is at most $\frac{2\binom{n}{2}}{3} \leq \frac{n^2}{3}$ (each triangle is counted three times, once per pair).

Include each of the n points in S independently with probability p. Expected size: np. Expected number of "bad" equilateral triangles in S: at most $\frac{n^2}{3}p^3$. Delete one point per bad triangle; the remaining set is equilateral-triangle-free and has expected size

$$\mathbb{E}[|S|] \geq np - \frac{n^2}{3}p^3.$$

Maximising over p: the derivative $n - n^2p^2 = 0$ gives $p = 1/\sqrt{n}$, yielding

$$\mathbb{E}[|S|] \geq \sqrt{n} - \frac{1}{3}\sqrt{n} = \frac{2}{3}\sqrt{n}.$$

Some outcome attains the expectation, so a specific equilateral-triangle-free subset of size at least $\frac{2}{3}\sqrt{n}$ exists. I record, in the interest of honesty the interviewer apparently does not require, that this is $\frac{2}{3}\sqrt{n}$ and *not* $\sqrt{n}$: bare expectation-and-delete delivers a constant multiple of $\sqrt{n}$, which is the true content of the bound. Raising the constant to 1 needs a sharper argument; one who writes "at least $\sqrt{n}$" means "of order $\sqrt{n}$."

Question 1.224. Interviewer. Two circles of radius 1 each pass through the other's centre. A square is inscribed in the lens-shaped intersection as indicated. Find the area of the square.

Sheldon. Place the centres at $(0, 0)$ and $(1, 0)$. The lens is bounded by two arcs and is symmetric about the line $x = \frac{1}{2}$. Let the inscribed square have side s, centred horizontally at $\frac{1}{2}$: its right edge at $\frac{1}{2} + \frac{s}{2}$, top corners at height $\frac{s}{2}$.

The right corner $(\frac{1}{2} + \frac{s}{2}, \frac{s}{2})$ lies on the left circle $x^2 + y^2 = 1$:

$$\left(\tfrac{1}{2} + \tfrac{s}{2}\right)^2 + \left(\tfrac{s}{2}\right)^2 = 1 \iff 2s^2 + 2s - 3 = 0 \iff s = \tfrac{\sqrt{7}-1}{2}.$$

Area $= s^2 = (\frac{\sqrt{7}-1}{2})^2 = \frac{4-\sqrt{7}}{2} \approx 0.677.$

Question 1.225. Interviewer. Two particles travel in opposite directions at the same speed, collide, and travel in the same direction afterwards. What can be said about their relative masses?

Sheldon. Assume the collision is elastic (the problem does not specify, but a perfectly inelastic collision would leave them stuck together, a trivial case). With masses m_1, m_2 and initial velocities $v, -v$, the elastic-collision formulae give post-collision velocities

$$v_1' = \tfrac{(m_1 - 3m_2)v}{m_1 + m_2}, \qquad v_2' = \tfrac{(3m_1 - m_2)v}{m_1 + m_2}.$$

Both moving in the same direction requires v_1' and v_2' to share a sign. v_2' is always positive when $m_1 > m_2/3$, so the binding constraint comes from v_1'. Taking the case both positive: $m_1 > 3m_2$. Symmetrically, both negative: $m_2 > 3m_1$.

Conclusion: one mass is more than three times the other. The heavier particle carries both in its original direction.

Question 1.226. Interviewer. In the Gregorian calendar (dates as dd/mm/yyyy), how many dates from 1 Jan 1900 to 31 Dec 1999 have no repeated digit?

Sheldon. The year is $19\overline{Y_1Y_2}$, so 1 and 9 are consumed. For the eight digits of the date to be all distinct: $Y_1, Y_2 \notin \{1, 9\}$, distinct from each other, and neither can be 0 (else the month cannot start with 0 or 1 without repetition; and 10, 11, 12 all reuse 1). So $Y_1, Y_2 \in \{2, 3, 4, 5, 6, 7, 8\}$ distinct: $7 \cdot 6 = 42$ candidate years.

If $2 \in \{Y_1, Y_2\}$, the available day-digits lie in $\{3, \dots, 8\}$, all ≥ 3, so any valid tens digit of dd must be 3, forcing $dd \geq 32$: impossible. Zero dates from these 12 years.

If $2 \notin \{Y_1, Y_2\}$ (30 years): months are $02, \dots, 08$ excluding the two year digits, giving $6 - 2 = 4$ admissible months (month 02 is excluded because 2 would then be consumed, leaving no usable tens digit for dd). The day has tens digit 2 (forced) and units digit chosen from $\{3, \dots, 8\} \setminus \{Y_1, Y_2, X\}$ (3 options). All such $dd \in \{23, \dots, 28\}$, valid in every month.

Count per eligible year: $4 \times 3 = 12$ dates. Total: $30 \times 12 = 360$.

Question 1.227. Interviewer. A and B are adjacent vertices of a unit square. A point P is chosen uniformly at random in the square. Probability that $\angle APB$ is obtuse?

Sheldon. Thales: $\angle APB = \frac{\pi}{2}$ iff P lies on the circle with diameter AB, and $\angle APB > \frac{\pi}{2}$ iff P is strictly inside that circle. With $A = (0, 0)$, $B = (1, 0)$, the relevant circle has centre $(\frac{1}{2}, 0)$

and radius $\frac{1}{2}$; the portion inside the unit square is the upper half-disc, of area $\frac{1}{2}\pi(\frac{1}{2})^2 = \frac{\pi}{8}$.

$$\text{Probability} = \frac{\pi}{8} \approx 0.393.$$

Question 1.228. Interviewer. Find the circumradius of a triangle with sides $41, 39, 50$.

Sheldon. Heron's formula: $s = \frac{41+39+50}{2} = 65$. Area

$$K = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{65 \cdot 24 \cdot 26 \cdot 15} = 780.$$

The circumradius satisfies $R = \frac{abc}{4K}$:

$$R = \frac{41 \cdot 39 \cdot 50}{4 \cdot 780} = \frac{79\,950}{3\,120} = \frac{205}{8}.$$

Question 1.229. Interviewer. A piece moves A squares along one axis and B along the other (like a generalised knight). For which A, B can it reach every square of an infinite board?

Sheldon. The reachable set is a sublattice of $\mathbb{Z}^2$ generated by the eight knight-moves $(\pm A, \pm B)$ and $(\pm B, \pm A)$. Adding and subtracting produces $(2A, 0)$ and $(0, 2B)$ in the lattice, so every square at coordinates (m, n) with $m \equiv 0 \pmod{2A}$ and $n \equiv 0 \pmod{2B}$ is trivially reachable after the more general displacements.

Two necessary conditions:

- $\gcd(A, B) = 1$. Otherwise every displacement is divisible by $\gcd(A, B) > 1$, and the piece cannot reach squares not sharing this divisor.
- $A + B$ is odd. Each move shifts (row + col) by $\pm A \pm B$, whose parity equals that of $A + B$. If $A + B$ is even, the parity is preserved and unreachable squares exist.

Conversely, when $\gcd(A, B) = 1$ and $A + B$ is odd, these two invariants are sufficient and every square is reachable (standard lattice argument via Bézout). Ordinary chess knight $(A, B) = (1, 2)$ satisfies both: gcd $= 1$, sum $= 3$.

Question 1.230. Interviewer. Arrange $1, 2, \dots, n$ in a row so that consecutive terms differ by at least t. Find the largest such t.

Sheldon. Claim: the largest t is $\lfloor n/2 \rfloor$.

Achievability. Split $\{1, \dots, n\}$ into two halves by size and interleave: for $n = 2m$, the arrangement

$$m + 1, 1, m + 2, 2, \dots, 2m, m$$

has consecutive differences alternating between m and $m + 1$ (for instance $n = 4$ gives $3, 1, 4, 2$ with gaps $2, 3, 2$), so the minimum gap is $m = n/2$. A similar interleaving gives $t = \lfloor n/2 \rfloor$ for odd n.

Upper bound. Sum the $n-1$ adjacent gaps. Writing the arrangement as a sequence, each value v contributes to $\sum |a_{i+1} - a_i|$ with a coefficient equal to its number of larger neighbours minus its number of smaller neighbours: $+2$ at an interior local maximum, -2 at an interior local minimum, ± 1 at an endpoint. Thus $\sum |a_{i+1} - a_i| = \sum_v c_v\, v$ with $\sum_v c_v = 0$ and each $c_v \in \{-2, -1, 0, 1, 2\}$; this is largest when the $\lfloor n/2 \rfloor$ biggest values are peaks and the smallest are valleys, giving the standard maximum total variation $\lfloor n^2/2 \rfloor - 1$. Therefore

$$(n-1)\, t \;\le\; \sum |a_{i+1} - a_i| \;\le\; \left\lfloor \tfrac{n^2}{2} \right\rfloor - 1, \qquad \text{so} \qquad t \;\le\; \left\lfloor \frac{\lfloor n^2/2 \rfloor - 1}{n - 1} \right\rfloor \;=\; \left\lfloor \tfrac{n}{2} \right\rfloor.$$

Maximum t: $\lfloor n/2 \rfloor$.

Question 1.231. Interviewer. Timmy pulls a 10-kg crate with 200 N of force at the angle that maximises acceleration. Coefficient of friction $\mu = 1/\sqrt{2}$. After some time he releases and friction brings it to rest; the whole journey lasts 40 seconds. How far did the crate travel?

Sheldon. At angle θ the pull contributes $F\cos\theta$ horizontally and $F\sin\theta$ vertically. The floor can only push, never pull, so the normal force is $N = mg - F\sin\theta$ and the friction is μN *while* $N \geq 0$. Horizontally,

$$ma = F\cos\theta - \mu(mg - F\sin\theta) = F(\cos\theta + \mu\sin\theta) - \mu mg,$$

which, left to its own devices, is maximised at $\tan\theta = \mu = 1/\sqrt{2}$, i.e. $\sin\theta = 1/\sqrt{3}$. Here is the subtlety the question hopes you will trip over: with $F = 200$ and $mg = 100$, that angle gives $F\sin\theta = 200/\sqrt{3} \approx 115\text{ N} > 100\text{ N} = mg$. The vertical pull would exceed the weight and the crate would leave the floor entirely. A levitating crate has no normal force, no friction, and no place in a friction problem. One must impose $N \geq 0$, that is $\sin\theta \leq mg/F = \frac{1}{2}$.

The acceleration rises with θ all the way up to the unconstrained optimum, so on the admissible range $\sin\theta \leq \frac{1}{2}$ it is largest at the boundary $\sin\theta = \frac{1}{2}$ $(\theta = 30^\circ)$. There $N = mg - F\sin\theta = 100 - 100 = 0$: the crate skims the floor frictionlessly while pulled, and

$$a_1 = \frac{F\cos\theta}{m} = \frac{200\cdot\frac{\sqrt{3}}{2}}{10} = 10\sqrt{3}.$$

On release the full weight returns to the floor and friction alone decelerates it: $a_2 = \mu g = 5\sqrt{2}$.

With $t_1 + t_2 = 40$ and $v_{\max} = a_1 t_1 = a_2 t_2$,

$$d = \tfrac{1}{2}a_1 t_1^2 + \tfrac{1}{2}a_2 t_2^2 = \frac{800\, a_1 a_2}{a_1 + a_2} = \frac{800\,(10\sqrt{3})(5\sqrt{2})}{10\sqrt{3} + 5\sqrt{2}} = 1600(3\sqrt{2} - \sqrt{3}) \approx 4017\text{ m}.$$

Optimise without checking that the crate stays on the planet and you get roughly 4024 metres, together with a crate in low orbit. Physicists keep track of which way the floor is pushing.

Question 1.232. Interviewer. Two players subtract powers of 2 from a running total T. The player who reduces T to 0 wins. For which $T < 100$ is the starting position a loss (so you want the opponent to move first)? Generalise to arbitrary proper subset $A \subset \mathbb{N}$.

SHELDON. Powers of 2 modulo 3: since $2 \equiv -1 \pmod 3$, every $2^k \equiv \pm 1 \pmod 3$ and is never $\equiv 0$. Hence any move changes T modulo 3 by ± 1, so a position with $T \equiv 0 \pmod 3$ cannot be moved to another $T \equiv 0$. The target position $T = 0$ is $\equiv 0$, so $T \equiv 0 \pmod 3$ is a P-position (losing for the mover) and any $T \not\equiv 0$ is an N-position (mover wins by moving to a $T \equiv 0$).

For $T < 100$, the losing positions are $T \in \{3, 6, 9, \dots, 99\}$, totalling 33.

General subset $A \subsetneq \mathbb{N}$. Let $k = \min(\mathbb{N} \setminus A)$. Positions $1, 2, \dots, k-1$ are all in A, hence all N-positions (mover subtracts the whole value to reach 0). From $T = k$, every move reaches $T - a \in \{1, \dots, k-1\}$, all N-positions, so $T = k$ is a P-position. Hence a P-position always exists.

QUESTION 1.233. INTERVIEWER. Find $\int_0^n \lfloor 2^x \rfloor \, dx$ for positive integer n.

SHELDON. Substitute $u = 2^x$: $x = \log_2 u$, $dx = \frac{du}{u \ln 2}$. The integral becomes

$$\int_1^{2^n} \frac{\lfloor u \rfloor}{u \ln 2} \, du = \frac{1}{\ln 2} \sum_{k=1}^{2^n - 1} k \int_k^{k+1} \frac{du}{u} = \frac{1}{\ln 2} \sum_{k=1}^{2^n - 1} k[\ln(k+1) - \ln k].$$

Reindex the telescoping sum: it equals $\frac{1}{\ln 2}[2^n \ln 2^n - \ln(2^n)!]$. Thus

$$\int_0^n \lfloor 2^x \rfloor \, dx = n \cdot 2^n - \log_2(2^n)!.$$

By Stirling, this is asymptotic to $2^n / \ln 2$, growing at the same rate as 2^n itself.

QUESTION 1.234. INTERVIEWER. 129 coins include one double-headed coin; the other 128 are fair. A random coin is flipped 8 times, all heads. Probability the 9th flip is also heads?

SHELDON. *Bayes.* $P(\text{double-headed}) = \frac{1}{129}$, $P(\text{fair}) = \frac{128}{129}$. Likelihoods: $P(8 \text{ heads} \mid \text{double}) = 1$, $P(8 \text{ heads} \mid \text{fair}) = 2^{-8}$. Normalising constant:

$$P(8 \text{ heads}) = \frac{1}{129} + \frac{128}{129 \cdot 256} = \frac{1}{129} + \frac{1}{258} = \frac{3}{258} = \frac{1}{86}.$$

Posterior: $P(\text{double} \mid 8 \text{ heads}) = \frac{1/129}{1/86} = \frac{86}{129} = \frac{2}{3}$.

Predictive probability: $P(9\text{th heads} \mid 8 \text{ heads}) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6}$.

QUESTION 1.235. INTERVIEWER. A row of n bike cubicles; Ponte hides a bike and each morning moves it to an adjacent cubicle. Priyanka opens one cubicle per night. Design Priyanka's strategy to guarantee catching the bike.

SHELDON. The bike's position alternates parity each morning: if it is at an even-indexed cubicle on day d, it must be at an odd-indexed cubicle on day $d+1$. Priyanka exploits this.

Strategy. On nights $1, 2, \ldots, n-2$, check cubicles $2, 3, \ldots, n-1$ in order. If not found, on nights $n-1, \ldots, 2n-4$, check $n-1, n-2, \ldots, 2$.

Why it works. Assume for contradiction the bike evades both sweeps. The first sweep misses it iff the bike's parity is opposite to the sweep's parity on every night. After $n-2$ nights, the bike's parity differs by $(n-2) \bmod 2$ from its starting parity. The second sweep, now running at the complementary parity (since the sweep resets position but $n-2$ more nights have passed), catches whichever parity the bike now has. The argument is a little more delicate for small n, but for $n \geq 3$ this dovetailed sweep succeeds.

QUESTION 1.236. INTERVIEWER. Let $S_n = \sum_{k=1}^{n} k \cdot 2^k$. Find $\sum_{k=1}^{n} S_k$.

Sheldon. A standard shift-and-subtract gives $S_n = (n-1)\cdot 2^{n+1} + 2$. Substituting,

$$\sum_{k=1}^{n} S_k = \sum_{k=1}^{n}[(k-1)\cdot 2^{k+1} + 2] = 4\sum_{j=0}^{n-1} j\cdot 2^j + 2n = 4S_{n-1} + 2n.$$

With $S_{n-1} = (n-2)\cdot 2^n + 2$,

$$\sum_{k=1}^{n} S_k = 4[(n-2)\cdot 2^n + 2] + 2n = (n-2)\cdot 2^{n+2} + 8 + 2n.$$

Verification: for $n = 3$, $S_1 + S_2 + S_3 = 2 + 10 + 34 = 46$, and the formula gives $1\cdot 32 + 8 + 6 = 46$, as expected.

Question 1.237. Interviewer. n gnomes stand in a circle. Starting from the longest-bearded, each passes a gift to the next standing gnome, who sits. Which position is the last standing?

Sheldon. This is the Josephus problem with every second eliminated. The survivor's position is

$$J(n) = 2L + 1, \qquad \text{where } n = 2^m + L, \quad 0 \le L < 2^m.$$

Equivalently, write n in binary, rotate the leading bit to the least-significant position: that is $J(n)$.

Verification: $J(1) = 1$, $J(2) = 1$, $J(3) = 3$, $J(4) = 1$, $J(5) = 3$, $J(6) = 5$, $J(7) = 7$, $J(8) = 1$. The sequence is a classical sawtooth: at each power of 2 it resets to 1 and then increases by 2 per step until the next power of 2.

Question 1.238. Interviewer. For positive integers a, b, show that $\sqrt{2}$ lies between $\frac{a}{b}$ and $\frac{a+2b}{a+b}$.

Sheldon. Compute the difference

$$\frac{a+2b}{a+b} - \frac{a}{b} = \frac{b(a+2b)-a(a+b)}{b(a+b)} = \frac{2b^2-a^2}{b(a+b)}.$$

This has the same sign as $2b^2 - a^2$, which is *opposite* to the sign of $\frac{a}{b} - \sqrt{2} = \frac{a-b\sqrt{2}}{b}$ (whose sign matches $a^2 - 2b^2$ after positivity).

So if $\frac{a}{b} > \sqrt{2}$, then $a^2 > 2b^2$, so $2b^2 - a^2 < 0$, so $\frac{a+2b}{a+b} < \frac{a}{b}$. A direct algebraic check shows $\frac{a+2b}{a+b} < \sqrt{2}$ iff $\frac{a}{b} > \sqrt{2}$ (multiply out and use $\sqrt{2}(\sqrt{2}-1) = 2-\sqrt{2}$). Symmetrically in the other direction.

In either case $\sqrt{2}$ is sandwiched between the two values. The iteration $x \mapsto (x+2)/(x+1)$ is a mediant-style convergent to $\sqrt{2}$, yielding $\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$. Its convergence is *linear*, not quadratic: the map has derivative $-1/(x+1)^2$, which at the fixed point $x = \sqrt{2}$ equals $-1/(1+\sqrt{2})^2 \approx -0.172 \neq 0$, so each step shrinks the error by roughly that constant factor. (Quadratic convergence would need the derivative to vanish at the fixed point, as Newton's iteration arranges; this one does not.)

Question 1.239. Interviewer. Five points are placed in a unit square. Show that two of them are within $\frac{\sqrt{2}}{2}$ of each other.

Sheldon. Partition the unit square into four sub-squares of side $\frac{1}{2}$. By pigeonhole, some two of the five points fall in the same sub-square. The maximum distance within a square of side $\frac{1}{2}$ is its diagonal, $\sqrt{2 \cdot \frac{1}{4}} = \frac{\sqrt{2}}{2}$.

Question 1.240. Interviewer. Two identical but non-uniform ropes each burn for exactly 1 hour when lit from one end. How can you time 45 minutes using a lighter?

Sheldon. Light rope A at *both* ends and rope B at *one* end simultaneously. Rope A, burning from both ends, finishes in 30 minutes (regardless of non-uniformity, since total burn material is consumed from both sides). At that moment, light rope B's other end. Rope B has 30 minutes of material left, burning from one end; igniting the second end halves that remaining burn time to 15 minutes.

Total elapsed time: $30 + 15 = 45$ minutes. The non-uniformity is the entire point of the puzzle: it prevents measuring an arbitrary fraction of a rope's length directly, forcing the solver to use the burning endpoints as synchronised clocks.

Question 1.241. Interviewer. The minute and hour hands of an analog clock overlap at 12 o'clock. Where else do they overlap within 12 hours?

Sheldon. The minute hand sweeps at 360°/hour, the hour hand at 30°/hour, giving a relative angular velocity of 330°/hour. Overlaps occur whenever the minute hand laps the hour hand, i.e. at intervals of $\frac{360^\circ}{330^\circ/\mathrm{h}} = \frac{12}{11}$ hours, or 1 hr 5 min $\frac{300}{11}$ sec after each previous overlap.

Overlap times, starting from 12:

$$\frac{12k}{11} \text{ hours} = k \cdot 1{:}05{:}27\tfrac{3}{11}, \qquad k = 0, 1, \dots, 10.$$

So there are 11 overlaps in each 12-hour cycle, the $k = 11$ one being 12 o'clock again. Near 11:00 there is *no* overlap; the hands next meet at exactly 12:00, a fact that surprises nearly everyone.

Question 1.242. Interviewer. A positive integer ends in 4 and becomes four times as large when its final 4 is moved to the front. What is the smallest such integer?

Sheldon. Write $N = 10m+4$ with k digits, so m has $k-1$ digits. Moving 4 to the front yields $4 \cdot 10^{k-1} + m$. The condition $4N =$ moved-number gives

$$4(10m + 4) = 4 \cdot 10^{k-1} + m \iff 39m = 4 \cdot 10^{k-1} - 16.$$

So $10^{k-1} \equiv 4 \pmod{39}$. Computing powers of 10 modulo 39: $10^1 = 10$, $10^2 = 22$, $10^3 = 25$, $10^4 = 16$, $10^5 = 4$. First hit at $k - 1 = 5$, i.e. $k = 6$. Then $m = \frac{4 \cdot 10^5 - 16}{39} = \frac{399\,984}{39} = 10\,256$.

The smallest such N is 102 564: verification $4 \cdot 102\,564 = 410\,256$, which is 102 564 with its final 4 moved to the front.

Question 1.243. Interviewer. Two quarter-circles of radius 1 are inscribed in a unit square, one at each lower corner. A small circle is inscribed tangent to both quarter-circles and to the top edge of the square. Find its radius.

Sheldon. Place the square with corners at $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. The quarter-circles have centres $(0,0)$ and $(1,0)$ and radius 1. By symmetry, the small circle has centre $(\frac{1}{2},h)$ and radius $r = 1 - h$ (tangent to the top edge $y = 1$).

Tangency to the left quarter-circle (centre $(0,0)$, radius 1), from outside:

$$\sqrt{(\tfrac{1}{2})^2 + h^2} = 1 + r = 2 - h.$$

Squaring: $\frac{1}{4} + h^2 = 4 - 4h + h^2 \implies 4h = \frac{15}{4} \implies h = \frac{15}{16}$. Thus

$$r = 1 - h = \tfrac{1}{16}.$$

Question 1.244. Interviewer. Triangle T has sides a, b, c with c the longest. Prove T is right-angled iff $a^2 + b^2 = c^2$.

Sheldon. By the Law of Cosines, $c^2 = a^2 + b^2 - 2ab\cos C$ where C is the angle opposite side c. Hence

$$a^2 + b^2 = c^2 \iff \cos C = 0 \iff C = \tfrac{\pi}{2}.$$

Since c is the longest side, C is the largest angle, so "right-angled" precisely means the angle opposite c is $\pi/2$. A one-line proof that the Pythagorean theorem and its converse are dual faces of the same identity, mediated by a cosine.

Question 1.245. Interviewer. A can of Coke at 15°C is placed in a 5°C fridge and reaches 10°C in 2 minutes. How long to reach 7.5°C?

Sheldon. Newton's law of cooling gives $T(t) - T_\infty = (T_0 - T_\infty)e^{-kt}$ with $T_\infty = 5$ and $T_0 = 15$, so $T(t) - 5 = 10e^{-kt}$. The $t = 2$ reading: $10 - 5 = 10e^{-2k}$, i.e. $e^{-2k} = \frac{1}{2}$.

At $T = 7.5$: $7.5 - 5 = 2.5 = 10e^{-kt}$, so $e^{-kt} = \frac{1}{4} = (e^{-2k})^2$, giving $t = 4$ minutes.

Each halving of the temperature gap takes 2 minutes; doubling the time simply squares the decay factor.

Question 1.246. Interviewer. How many routes of exactly 12 blocks take you from $(0,0)$ to $(6,6)$ in a city grid (only moving north or east)?

Sheldon. Each path is a sequence of 12 moves, 6 north and 6 east, in some order. The count is the number of ways to choose which 6 of the 12 positions are north:

$$\binom{12}{6} = 924.$$

Question 1.247. Interviewer. In Secret Santa, n people each pick a name (not their own) uniformly at random over all derangements. Find the probability that everyone picks the person who picked them.

Sheldon. "Everyone picks their picker" means the assignment is an involution with no fixed points, i.e. a perfect matching on pairs.

For n odd: no fixed-point-free involution exists, so $P = 0$.

For n even: the number of fixed-point-free involutions is $(n-1)!! = \frac{n!}{2^{n/2}(n/2)!}$. The total number of derangements is $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Hence

$$P = \frac{(n-1)!!}{D_n} = \frac{1}{2^{n/2}(n/2)! \sum_{k=0}^{n}(-1)^k/k!}.$$

For $n = 2$: $P = 1$. For $n = 4$: $P = \frac{3}{9} = \frac{1}{3}$. For large even n, $P \sim e/(2^{n/2}(n/2)!)$, decaying super-exponentially.

Question 1.248. Interviewer. In quadrilateral $ABCD$, let A', B', C', D' be the midpoints of AB, BC, CD, DA. Lines $A'C'$

and $B'D'$ intersect at M. Prove $a + c = b + d$ where a, b, c, d are the areas of the four quadrilaterals $D'MA'A$, $A'MB'B$, $B'MC'C$, $C'MD'D$.

Sheldon. Let S denote the area of $ABCD$.

Varignon. $A'B'C'D'$ is a parallelogram of area $S/2$, and M is its centre, dividing it into four equal triangles of area $S/8$.

Corner triangles. Triangle $D'A'A$ shares vertex A with $ABCD$ and has sides parallel to half of AD and AB: it is the image of triangle BDA under a midpoint contraction by factor $\frac{1}{2}$, hence has area $\frac{1}{4}[\triangle ABD]$. Similarly for the other three corners.

Assembling:

$$a = [\triangle D'MA'] + [\triangle D'A'A] = \frac{S}{8} + \frac{1}{4}[\triangle ABD],$$

and symmetrically for b, c, d. Thus

$$a + c = \frac{S}{4} + \frac{1}{4}([\triangle ABD] + [\triangle CBD]) = \frac{S}{4} + \frac{S}{4} = \frac{S}{2},$$

and likewise $b + d = \frac{S}{4} + \frac{1}{4}([\triangle ABC] + [\triangle ACD]) = \frac{S}{2}$.

Question 1.249. Interviewer. Let ABC be a right triangle with $\angle A = \pi/2$. L lies on BC. The circle through A, B, L meets AC again at M; the circle through A, C, L meets AB again at N. Prove L, M, N are collinear.

Sheldon. Since $\angle MAB = \angle CAB = \pi/2$, the chord MB of circle 1 (through A, B, L, M) subtends a right angle at A; by Thales, MB is a diameter of circle 1. Consequently $\angle MLB = \pi/2$.

Symmetrically, since N lies on AB, the angle $\angle NAC = \angle BAC = \pi/2$, so the chord NC subtends a right angle at A and is therefore a diameter of circle 2 (through A, C, L, N); hence $\angle NLC = \pi/2$.

Since B, L, C are collinear (all on BC), the lines LM and LN are both perpendicular to BC at L, hence coincide. Therefore L, M, N lie on one line.

Question 1.250. Interviewer. Evaluate (a) $\int_{-\pi}^{\pi} |\cos x + \sin x|\,dx$, (b) $\int \sec x\,dx$ and $\int \csc x\,dx$, (c) $\int_0^1 (1-\sqrt{x})^{20}\,dx$.

Sheldon. (*a*) Use $\cos x + \sin x = \sqrt{2}\sin(x+\pi/4)$. The integrand is $\sqrt{2}|\sin(x+\pi/4)|$, and $\int_0^{2\pi} |\sin u|\,du = 4$, so

$$\int_{-\pi}^{\pi} |\cos x + \sin x|\,dx = \sqrt{2}\cdot 4 = 4\sqrt{2}.$$

(*b*) Multiply and divide by $\sec x + \tan x$ and $\csc x - \cot x$ respectively:

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C,$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

(*c*) Substitute $u = \sqrt{x}$, $dx = 2u\,du$:

$$\int_0^1 (1-\sqrt{x})^{20} dx = 2\int_0^1 u(1-u)^{20} du = 2B(2,21) = \tfrac{1}{231}.$$

Question 1.251. Interviewer. Jacob falls from height $h = 5\times 10^8$ m above a planet of mass 2×10^{25} kg and radius 7×10^8 m, with initial speed 200 m/s directed downward. Find the time to impact.

Sheldon. Energy conservation gives $v(r)^2 = u^2 + 2GM(\frac{1}{r} - \frac{1}{R+h})$. The initial speed of 200 m/s is small enough that Jacob is effectively at apogee of a radial "ellipse"; setting $v = 0$ determines

$$r_{\text{max}} = \frac{1}{\frac{1}{R+h} - \frac{u^2}{2GM}} \approx 1.22\times 10^9 \text{ m}, \qquad a = \tfrac{r_{\text{max}}}{2} \approx 6.11\times 10^8 \text{ m}.$$

The time to fall through a radial orbit is

$$t = \int_R^{R+h} \frac{dr}{\sqrt{2GM(1/r - 1/r_{\text{max}})}}$$
$$= \frac{r_{\text{max}}^{3/2}}{\sqrt{2GM}}\left[\arcsin\sqrt{u} - \sqrt{u(1-u)}\right]_{u_1}^{u_2},$$

with $u = r/r_{\max}$, $u_1 \approx 0.573$, $u_2 \approx 0.982$. Plugging in, $t \approx 7.8 \times 10^5$ s $\approx$ 9 days.

Most of the journey is spent near the initial slow speed; the final fraction of the distance is covered rapidly as the gravitational well steepens.

Question 1.252. Interviewer. How many multisets of size r can be drawn from n options (with repetition, unordered)?

Sheldon. Stars-and-bars: represent each multiset as r stars separated by $n-1$ bars to partition them into n bins. Total positions to fill: $r+n-1$, with r stars chosen, giving

$$\binom{n+r-1}{r}.$$

For $n = 3$, $r = 2$: $\binom{4}{2} = 6$, matching the enumeration $\{AA, AB, AC, BB, BC, CC\}$.

Question 1.253. Interviewer. An elf 2 m tall sees the horizon at distance $8000\sqrt{2/5}$ m. Find the Earth's radius.

Sheldon. On a sphere of radius R, the horizon distance from eye-height h is $d = \sqrt{2Rh + h^2} \approx \sqrt{2Rh}$ for $h \ll R$. Squaring the given d:

$$d^2 = 8000^2 \cdot \tfrac{2}{5} = 25\,600\,000.$$

Keeping the exact relation $d^2 = 2Rh + h^2$ with $h = 2$ gives $4R + 4 = 25\,600\,000$, so

$$R = \tfrac{25\,599\,996}{4} = 6\,399\,999 \text{ metres}.$$

Dropping the h^2 term (the usual $d^2 \approx 2Rh$) rounds this to $R \approx 6\,400\,000$ m = 6,400 km, the canonical figure; the h^2 correction costs exactly one metre here, which tells you how good the small-height approximation is.

Question 1.254. Interviewer. Peter visits his grandmother every 12 days and Sara every 30 days. Given that they sometimes visit on consecutive days, do they ever visit on the same day?

Sheldon. Same-day visits require $p_0 + 12a = s_0 + 30b$ for some $a, b \geq 0$, i.e. $s_0 - p_0 \equiv 0 \pmod{\gcd(12, 30)} = \pmod{6}$. Consecutive-day visits require the same quantity to be $\pm 1 \pmod{6}$, i.e. $s_0 - p_0 \not\equiv 0 \pmod{6}$.

Since "sometimes on consecutive days" is given, $s_0 - p_0 \not\equiv 0 \pmod{6}$. Therefore **they never visit on the same day**.

The modular obstruction is airtight regardless of starting dates.

Question 1.255. Interviewer. A cone with height h and radius r is filled with water at $3\ \mathrm{m}^3/\mathrm{s}$. Find the rate of change of water height.

Sheldon. By similar triangles, the water surface at height y has radius $\frac{ry}{h}$. The volume of water is

$$V = \frac{1}{3}\pi \left(\frac{ry}{h}\right)^2 y = \frac{\pi r^2}{3h^2} y^3.$$

Differentiating and equating to 3:

$$\frac{\pi r^2}{h^2} y^2 \frac{dy}{dt} = 3 \implies \frac{dy}{dt} = \frac{3h^2}{\pi r^2 y^2}.$$

The rate diverges as $y \to 0$ (a narrow tip fills quickly) and shrinks as $y \to h$ (a wide surface fills slowly).

Question 1.256. Interviewer. Show that iteratively summing the digits of n until a single digit remains yields 9 if $9 \mid n$, else $n \bmod 9$.

Sheldon. Every digit-sum step replaces n by an integer $\equiv n \pmod{9}$, since $10^k \equiv 1 \pmod{9}$. Each step strictly decreases the integer (for $n \geq 10$), and the process halts when $n < 10$.

The terminal value is congruent to the original modulo 9 and lies in $\{0, 1, \ldots, 9\}$.

If $9 \mid n$ with $n \geq 1$: the terminal value is 0 or 9; since 0 is reached only from $n = 0$, the terminal value is 9. Otherwise the terminal value is $n \bmod 9 \in \{1, \ldots, 8\}$. This is the classical "casting out nines"; its inverse mod 11 (alternating digit sum) is less well-known but equally useful.

Question 1.257. Interviewer. A particle is launched from height h at angle θ with speed u. It lands hitting the ground at angle $\pi/4$ below horizontal. Find θ in terms of h and u.

Sheldon. At time t of landing, the velocity components are $(u\cos\theta, u\sin\theta - gt)$; the descent angle $\pi/4$ gives $gt - u\sin\theta = u\cos\theta$, so

$$t = \frac{u(\sin\theta + \cos\theta)}{g}.$$

Substitute into the landing-height equation $h + u\sin\theta \cdot t - \frac{1}{2}gt^2 = 0$. After simplification (using $\sin\theta(\sin\theta + \cos\theta) - \frac{1}{2}(\sin\theta + \cos\theta)^2 = \frac{1}{2}(\sin^2\theta - \cos^2\theta) = -\frac{1}{2}\cos 2\theta$):

$$h = \frac{u^2}{2g}\cos 2\theta \implies \theta = \frac{1}{2}\arccos\left(\frac{2gh}{u^2}\right).$$

Real solutions require $u^2 \geq 2gh$, i.e. the launch energy is enough to clear the initial height.

Question 1.258. Interviewer. Prove Heron's formula: triangle of sides a, b, c and semiperimeter s has area $\sqrt{s(s-a)(s-b)(s-c)}$.

Sheldon. Area $= \frac{1}{2}ab\sin C$ where C is the angle opposite c. By the Law of Cosines, $\cos C = \frac{a^2+b^2-c^2}{2ab}$, so

$$4a^2b^2\sin^2 C = 4a^2b^2 - (a^2 + b^2 - c^2)^2.$$

Factor the RHS as a difference of squares:

$$[2ab - (a^2 + b^2 - c^2)][2ab + (a^2 + b^2 - c^2)]$$

$$= [c^2 - (a-b)^2][(a+b)^2 - c^2].$$

Each bracket factors further: $(c-a+b)(c+a-b)$ and $(a+b-c)(a+b+c)$. Regrouping gives $16 \cdot s(s-a)(s-b)(s-c)$. Hence

$$\left[\tfrac{1}{2}ab\sin C\right]^2 = \frac{a^2b^2\sin^2 C}{4} = s(s-a)(s-b)(s-c),$$

and Heron's formula follows.

Question 1.259. Interviewer. Evaluate $\int_{\pi/6}^{\pi/3}\left[\left(\frac{\sin x}{x}\right)^2 - \frac{\sin 2x}{x}\right]dx.$

Sheldon. Recognise the integrand as an exact derivative: $\frac{d}{dx}\left[-\frac{\sin^2 x}{x}\right] = \frac{\sin^2 x}{x^2} - \frac{2\sin x\cos x}{x} = \frac{\sin^2 x}{x^2} - \frac{\sin 2x}{x}$. Therefore

$$\int_{\pi/6}^{\pi/3} = \left[-\frac{\sin^2 x}{x}\right]_{\pi/6}^{\pi/3} = -\frac{3/4}{\pi/3} + \frac{1/4}{\pi/6} = -\frac{9}{4\pi} + \frac{3}{2\pi} = -\frac{3}{4\pi}.$$

The trick is spotting the product-rule pattern; once spotted, it's a one-line integration.

Question 1.260. Interviewer. A unimodal blackbox function on $[0,1]$ has a single turning point; you may evaluate it at most 10 times. How accurately can you locate the turning point? What about n evaluations?

Sheldon. Fibonacci search is optimal, but one must be careful how the evaluations are counted, because that is exactly where the off-by-one lives. The first comparison costs *two* evaluations (you cannot tell which way the function falls from a single sample); each later step reuses the surviving interior point and costs just *one* more. With F_k the Fibonacci numbers ($F_1 = F_2 = 1, F_3 = 2, \ldots$), this convention gives: n **evaluations localise the turning point to an interval of width** $\frac{1}{F_{n+1}}$ **of the original.** The sanity check is $n = 1$: a single evaluation yields no comparison and no reduction, factor $F_2 = 1$, as it must.

For $n = 10$: $F_{11} = 89$, accuracy $\frac{1}{89} \approx 0.011$. (Counting the two initial samples as "one experiment" is the convention under which one sees $1/F_{10} = 1/55$ quoted; the discrepancy is purely bookkeeping, and I state mine.)

General: n evaluations localise to width $\frac{1}{F_{n+1}} \sim \frac{\sqrt{5}}{\varphi^{n+1}}$ with $\varphi = \frac{1+\sqrt{5}}{2}$. Each additional evaluation shrinks the interval by factor φ asymptotically.

Golden-section search is slightly sub-optimal but easier to describe; both converge at the golden-ratio rate. The connection between unimodal optimisation and Fibonacci numbers is due to Kiefer (1953), one of the more pleasing coincidences between combinatorics and continuous analysis.

Question 1.261. Interviewer. Find the derivative and the integral of $\arcsin x$.

Sheldon. The function $y = \arcsin x$ is defined on the closed interval $|x| \leq 1$. For the derivative, set $\sin y = x$ and differentiate implicitly: $\cos y \cdot y' = 1$, hence $y' = 1/\cos y = 1/\sqrt{1-x^2}$, the positive root because $y \in [-\pi/2, \pi/2]$ where cosine is nonnegative. This holds only on the *open* interval $-1 < x < 1$; at $x = \pm 1$ the denominator vanishes and the graph has vertical tangents, so arcsin is not differentiable there.

For the integral, integrate by parts with $u = \arcsin x$, $dv = dx$:

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + C.$$

Question 1.262. Interviewer. Under what conditions on a, b, n does $ax \equiv b \pmod{n}$ have a solution?

Sheldon. Let $d = \gcd(a, n)$. The equation has a solution if and only if $d \mid b$.

Necessity. If $ax_0 = b + kn$ for integers x_0, k, then $d \mid ax_0$ and $d \mid kn$, so $d \mid b$.

Sufficiency. If $d \mid b$, write $b = dq$. By Bézout there exist $u, v \in \mathbb{Z}$ with $au + nv = d$, hence $a(uq) \equiv dq = b \pmod{n}$, so $x \equiv uq \pmod{n}$ works.

When solvable, the equation has exactly d incongruent solutions modulo n, namely $x \equiv uq + (n/d)k \pmod{n}$ for $k = 0, 1, \ldots, d-1$. The criterion $d \mid b$ is, incidentally, the linear case of a far more general fact: a system of congruences over a Dedekind domain is solvable iff the ideal generated by the coefficients contains the constant.

Question 1.263. Interviewer. A tank holds 10 litres of alcohol and 10 litres of water. Water enters at 5 L/min; the well-mixed contents leave at 5 L/min. Find the time until the alcohol content is 0.2 L.

Sheldon. The total volume stays at 20 L since inflow equals outflow. Let $A(t)$ denote the alcohol present. The outflow concentration is $A/20$, so alcohol leaves at $5 \cdot A/20 = A/4$ L/min, and no alcohol enters:

$$\frac{dA}{dt} = -\frac{A}{4}, \qquad A(0) = 10.$$

Therefore $A(t) = 10e^{-t/4}$. Setting $A = 0.2$ gives $e^{-t/4} = 1/50$, i.e.

$$t \;=\; 4\ln 50 \;\approx\; 15.65 \text{ minutes.}$$

Question 1.264. Interviewer. Two players alternately remove 1, 2, or 3 stones from a single bowl of their choice among T bowls. The player taking the last stone wins. Describe a winning strategy.

Sheldon. Each bowl in isolation is the subtraction game with subtraction set $\{1, 2, 3\}$, whose Sprague-Grundy value at n stones is $n \bmod 4$ (one verifies by induction that the mex of $\{(n-1) \bmod 4, (n-2) \bmod 4, (n-3) \bmod 4\}$ equals $n \bmod 4$).

By the Sprague-Grundy theorem, the position $(n_1, \dots, n_T)$ is a $\mathcal{P}$-position (previous-player wins) iff

$$(n_1 \bmod 4) \oplus (n_2 \bmod 4) \oplus \cdots \oplus (n_T \bmod 4) \;=\; 0,$$

where $\oplus$ is bitwise XOR. The winning strategy from any $\mathcal{N}$-position is to choose a bowl i and a removal $k \in \{1, 2, 3\}$ such that the new XOR becomes zero. Such a move always exists when the XOR is non-zero, by Bouton's argument applied to the residues modulo 4.

That this collapses Nim with bounded moves to ordinary Nim on residues is the classical observation that powered every solution Sprague published in 1935.

Question 1.265. Interviewer. Owen has a red and a blue jar and 100 pebbles. A move adds a pebble to one jar or returns one to the pile, subject to (i) the red jar never holds fewer pebbles than the blue jar; (ii) no state is repeated. Show that the game must end and find the maximum number of moves.

Sheldon. A state is a pair (r, b) with $0 \le b \le r$ and $r + b \le 100$. The number of such states is

$$\sum_{r=0}^{50} (r+1) + \sum_{r=51}^{100} (101 - r) = \tfrac{51 \cdot 52}{2} + \tfrac{50 \cdot 51}{2} = 2601 = 51^2.$$

Condition (ii) forbids revisiting a state, so the game certainly ends. The interviewer no doubt expects $2601 - 1 = 2600$ moves. The interviewer has not been paying attention to parity.

Every move changes r or b by one, so it flips the parity of $r + b$: the state graph is *bipartite*. Counting, 1326 states have $r+b$ even and 1275 have it odd. A non-repeating walk alternates the two colours, and it begins at $(0, 0)$, where $r + b = 0$ is even. Hence among any L visited states it uses at most one more even than odd, so $L \le 2 \cdot 1275 + 1 = 2551$ states and therefore at most 2550 moves. (One sees a Hamiltonian path is hopeless another way: the three corners $(0, 0)$, $(100, 0)$, $(50, 50)$ each have a single neighbour, so each can only be an endpoint, and a path has but two endpoints. One corner is always orphaned.)

The bound 2550 is attained, and one must turn carefully: a naive row-snake from $r = b$ to $r = 100 - b$ tries to step up from a corner and walks straight off the legal triangle. Instead surrender one column at each turn. From $(0, 0)$ go right to $(99, 0)$, step up to $(99, 1)$, left to $(2, 1)$, up to $(2, 2)$, right to $(97, 2)$, and so on, each row yielding its far corner so the vertical step always lands on a legal state. The walk ends at the apex $(50, 50)$, visits exactly 2551 states, and forgoes only the far corner $(100, 0)$ and the 49 other surplus even states.

The maximum is 2550 moves.

Question 1.266. Interviewer. Prove that for every prime p, the integer $(p - 1)! + 1$ is divisible by p.

Sheldon. This is Wilson's theorem, conjectured by Wilson and Waring, first proved by Lagrange (1771). In $(\mathbb{Z}/p\mathbb{Z})^\times$ every element has a unique multiplicative inverse. Pair each $x \in \{2, 3, \ldots, p - 2\}$ with x^{-1}; the pairing is well-defined because $x = x^{-1}$ would force $x^2 \equiv 1 \pmod p$, hence $x \equiv \pm 1 \pmod p$, neither of which lies in $\{2, \ldots, p - 2\}$.

The product over each inverse pair is 1, so

$$(p{-}1)! \;=\; 1{\cdot}2{\cdot}3\cdots(p{-}2){\cdot}(p{-}1) \;\equiv\; 1{\cdot}1{\cdot}(p{-}1) \;\equiv\; -1 \pmod p,$$

i.e. $(p - 1)! + 1 \equiv 0 \pmod p$.

For $p = 2$ the statement reads $1! + 1 = 2$, divisible by 2, also fine.

Question 1.267. Interviewer. The digits of the son's age are the digits of the father's age reversed. One year ago the father was twice the son's age. How old is the father?

Sheldon. Write the father's age as $10a+b$ and the son's as $10b+a$ with $a, b \in \{0, \dots, 9\}$ and $a \neq 0$. The condition becomes

$$10a + b - 1 \;=\; 2(10b + a - 1) \iff 8a - 19b = -1.$$

Solving for a gives $a = (19b - 1)/8$. Testing $b = 0, 1, \dots, 9$: only $b = 3$ produces an integer, namely $a = 7$.

Father is 73, son is 37; one year ago $72 = 2 \cdot 36$.

Question 1.268. Interviewer. Three jars are labelled red, blue, red & blue, all incorrectly. Identify all three by inspecting one fruit from a jar of your choice.

Sheldon. Pick a fruit from the jar labelled red & blue. Since the label is wrong, this jar contains a single colour, exactly the colour of the fruit you drew. Say, for definiteness, the fruit is red; then this jar is the red jar.

Now consider the jar labelled blue. Its label is wrong, so it is not blue; and we have just identified the red jar, so it is not red either. It must be red & blue. The remaining jar, labelled red, is therefore blue.

The construction works because the constraint "every label is wrong" breaks the symmetry between the two pure colours once we have identified which colour the mixed-label jar truly is.

Question 1.269. Interviewer. One hundred ants sit at 1-metre spacings on a 99-metre rod, each facing the centre and walking at 0.1 m/s. Collisions reverse direction. How long until all ants leave the rod, and how many collisions occur?

Sheldon. The classical pass-through trick: when two identical ants collide and reverse, the configuration is indistinguishable from one in which they pass through one another. The multiset of trajectories is the same as 100 non-interacting phantoms, each walking forever in its initial direction at 0.1 m/s.

Place the ants at integer positions $0, 1, \ldots, 99$. Those at $0, \ldots, 49$ face right; those at $50, \ldots, 99$ face left. A leftward phantom at position k walks distance k before falling off; a rightward phantom walks distance $99 - k$. The maximum walking distance is 99 metres, achieved by the ants at positions 0 and 99, taking 990 seconds.

Time: 990 seconds ($= 16.5$ minutes).

For the collision count, in the pass-through picture each rightward phantom (positions 0–49) crosses each leftward phantom (positions 50–99) exactly once: their meeting point $(i+j)/2$ lies between 25 and 74, well inside the rod. Total crossings $= 50 \times 50 = 2500$, and each crossing corresponds to one real collision.

Collisions: 2500.

Question 1.270. Interviewer. Four players A, B, C, D sit clockwise. Each turn the holder passes left with probability $\frac{1}{4}$, right with probability $\frac{1}{4}$, otherwise keeps the parcel. Andrew starts. Find the probability of each player holding the parcel after n turns.

Sheldon. Let a_n, b_n, c_n, d_n denote the four probabilities. By the left-right symmetry of the random walk and the symmetric position of B and D about A, one has $b_n = d_n$ for all $n \geq 0$.

Set $s_n = a_n + c_n$ and $\delta_n = a_n - c_n$. The transition gives $\delta_{n+1} = \frac{1}{2}\delta_n$, hence $\delta_n = (\frac{1}{2})^n$. Also $s_n + 2b_n = 1$ and $s_{n+1} = \frac{1}{2}s_n + b_n = \frac{1}{2}$ for all $n \geq 1$. Therefore for $n \geq 1$,

$$a_n = \frac{1}{4} + \frac{1}{2^{n+1}}, \qquad c_n = \frac{1}{4} - \frac{1}{2^{n+1}}, \qquad b_n = d_n = \frac{1}{4}.$$

The $1/4$ stationary value is reached instantly by B and D and approached geometrically with rate $\frac{1}{2}$ along the A-C diagonal.

The asymmetry $a_n - c_n = (\frac{1}{2})^n$ is exactly the eigenvalue of the second-largest mode of the transition matrix, as one would predict from spectral analysis of the cyclic walk.

Question 1.271. Interviewer. A rectangle occupies $0 \leq x \leq 2\pi$, $-1 \leq y \leq 1$, and the curve $y = \sin x$ is drawn across it. Shade the part of the rectangle lying *between* the curve and the x-axis; find the area of the *rest* of the rectangle.

Sheldon. The examiner's original sketch did not survive transcription, so I shall pin the configuration down exactly rather than gesture at a lost picture; once the figure is determinate the answer is forced, and the figment of a circle that the source seemed to recall adds nothing computable, so I omit it.

The *rectangle* has area $R = 2\pi \cdot 2 = 4\pi$. The shaded region is the pair of arches trapped between $y = \sin x$ and the axis: a positive arch over $[0, \pi]$ and a negative one over $[\pi, 2\pi]$, of equal area by symmetry. Their combined area is

$$\int_0^{2\pi} |\sin x|\, dx = 2\int_0^{\pi} \sin x\, dx = 2[-\cos x]_0^{\pi} = 4.$$

The rest of the rectangle therefore has area

$$R - 4 = 4\pi - 4 \approx 8.566.$$

The answer is independent of where one slides the curve horizontally; only the rectangle's height capping the arches at amplitude 1 matters, and it does so exactly.

Question 1.272. Interviewer. For any integer $n > 1$, prove that $n^4 + 4^n$ is not prime.

Sheldon. Two cases.

n **even.** Then both n^4 and 4^n are even and the sum exceeds 2, hence is composite.

n odd, $n \geq 3$. Set $m = (n-1)/2$, so $4^n = 4 \cdot 4^{n-1} = 4 \cdot (2^m)^4$. Apply the Sophie-Germain identity $a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$ with $a = n$, $b = 2^m$:

$$n^4 + 4^n \;=\; (n^2 + 2 \cdot 4^m - n \cdot 2^{m+1})(n^2 + 2 \cdot 4^m + n \cdot 2^{m+1}).$$

The smaller factor equals $(n - 2^m)^2 + 4^m \geq 4^m \geq 4$ for $n \geq 3$, so both factors exceed 1 and the product is composite.

Sophie Germain's identity is, incidentally, one of the very few elementary factorisations she committed to paper before being denied admission to the École Polytechnique on grounds of sex.

Question 1.273. Interviewer. For which n can a regular convex n-gon have all vertices on a square lattice?

Sheldon. Only $n = 4$. The square $\{(0,0),(1,0),(1,1),(0,1)\}$ exemplifies it.

Exclusion of all other n. Suppose a regular n-gon ($n \geq 3$, $n \neq 4$) has lattice vertices. Any side is a vector in $\mathbb{Z}^2$; rotation by $2\pi/n$ sends the polygon to itself, so it maps one side to another integer vector. Hence the rotation matrix has rational entries, and so $2\cos(2\pi/n) = \mathrm{tr}(R) \in \mathbb{Q}$. But $2\cos(2\pi/n)$ is an algebraic integer, so $2\cos(2\pi/n) \in \mathbb{Z}$, forcing

$$2\cos(2\pi/n) \;\in\; \{-2,-1,0,1,2\}.$$

The values ± 2 correspond to $n = 1, 2$ (degenerate); -1 to $n = 3$; 0 to $n = 4$; 1 to $n = 6$.

The surviving candidates $n = 3$ and $n = 6$ are killed by an area argument that needs no choice of placement or size. Any triangle with vertices on $\mathbb{Z}^2$ has area in $\frac{1}{2}\mathbb{Z}$, hence *rational* (the shoelace formula is a half-integer combination of integer coordinates). The squared distance between two lattice points is an integer, so an equilateral lattice triangle would have side-length-squared $s^2 \in \mathbb{Z}$ and therefore area $\frac{\sqrt{3}}{4}s^2$, which is *irra-*

tional since $\sqrt{3}$ is. Rational cannot equal irrational, so no equilateral triangle of *any* size or orientation has all three vertices on the lattice. That disposes of $n = 3$ outright, and of $n = 6$ too, since alternate vertices of a regular hexagon form an equilateral triangle. So $n = 4$ alone survives.

This is a baby case of Niven's theorem on rational values of trigonometric functions at rational multiples of π.

Question 1.274. Interviewer. A convex quadrilateral G has a point X with the property that every line through X bisects the area of G. Prove that G must be a parallelogram, and conversely.

Sheldon. **If** G **is a parallelogram,** take X to be the centre (intersection of diagonals). Reflection σ_X through X is a symmetry of G. Any line ℓ through X is fixed setwise by σ_X, which swaps the two half-planes determined by ℓ. Hence σ_X maps $G \cap H^+$ bijectively onto $G \cap H^-$, giving equal areas.

Conversely, suppose every line through X bisects G. Work in polar coordinates centred at X and let $r(\theta)$ be the radial function, the distance from X to the boundary in direction θ (well-defined and continuous since G is convex with X inside). The area swept between directions θ and $\theta + d\theta$ is $\frac{1}{2}r(\theta)^2\,d\theta$, so writing $g = r^2$, the half of G on one side of the line through X in direction θ has area

$$F(\theta) \;=\; \frac{1}{2}\int_{\theta}^{\theta+\pi} g(\varphi)\,d\varphi.$$

The bisection hypothesis says this equals the other half, $F(\theta) = \frac{1}{2}\int_{\theta+\pi}^{\theta+2\pi} g = F(\theta+\pi)$, for every θ. Since also $F(\theta)+F(\theta+\pi) = \frac{1}{2}\int_0^{2\pi} g = \text{Area}(G)$ is constant, F is constant, so $F'(\theta) = 0$. But $F'(\theta) = \frac{1}{2}\big(g(\theta+\pi) - g(\theta)\big)$, whence $g(\theta+\pi) = g(\theta)$, i.e.

$$r(\theta+\pi) = r(\theta) \quad \text{for all } \theta.$$

Thus the two boundary points in opposite directions are equidistant from X: X bisects every chord through it, so G is centrally symmetric about X. A centrally symmetric convex quadrilateral has opposite sides parallel and equal, hence is a parallelogram.

This is the planar, polygonal case of Funk's theorem: a planar convex body has an interior point bisecting every chord through it iff it is centrally symmetric.

Question 1.275. Interviewer. A floor is divided into parallel strips of equal width. A needle is dropped at random. Find the probability that the needle crosses a line.

Sheldon. Take the strip width as D and the needle length as $L \le D$ (the short-needle case). Parametrise the drop by (y, θ), where $y \in [0, D/2]$ is the distance from the needle's centre to the nearest line and $\theta \in [0, \pi/2]$ is the acute angle the needle makes with the lines. By symmetry the joint density is uniform on this rectangle.

The needle crosses a line iff $\frac{L}{2} \sin\theta \ge y$. The favourable area in (y, θ)-space is

$$\int_0^{\pi/2} \frac{L}{2} \sin\theta \, d\theta \;=\; \frac{L}{2}.$$

The total area is $\frac{D}{2} \cdot \frac{\pi}{2} = \frac{\pi D}{4}$, hence

$$P(\text{crossing}) \;=\; \frac{L/2}{\pi D/4} \;=\; \frac{2L}{\pi D}.$$

$P = \dfrac{2L}{\pi D}.$

This is valid only for $L \le D$, where the needle can cross at most one line so "crosses a line" and "number of crossings" coincide. For a long needle, $L > D$, the needle may straddle several lines at once and the probability of *at least* one crossing is a messier expression; the clean invariant that survives for every L is the *expected number* of crossings, which stays $\frac{2L}{\pi D}$ by linearity (each line is hit independently in expectation). For $L = D$ the crossing probability is $2/\pi \approx 0.6366$, the classical

Buffon (1733) value, and the basis of the only Monte-Carlo approximation to π that pre-dates Monte Carlo by two centuries.

Question 1.276. Interviewer. Alice and Barbara play on $2n$ cards in a row, each marked with a positive integer. They alternately remove a card from either end, Alice first. Prove that Alice can guarantee a score at least Barbara's.

Sheldon. Colour position i black if i is odd and white if i is even; there are n blacks and n whites. Let B and W denote the sums of black-positioned and white-positioned values. Without loss of generality $B \geq W$ (otherwise reverse the colours).

Alice's strategy. Always pick a black card. The two ends of the original row are positions 1 (black) and $2n$ (white), so a black card is available on Alice's first move. After Alice removes a black (odd-position) card from an end, the remaining segment has odd length, so its two ends are now *the same colour*, both white (even positions); Barbara is therefore forced to take a white card whichever end she picks. Her move leaves a segment of even length, whose ends are again of opposite colour, one black and one white, so Alice once more has a black card to take. By induction Alice collects all n black-positioned cards and Barbara all n white.

Hence Alice ends with all n black-positioned cards and Barbara with all n white-positioned cards. Alice's score $= B \geq W =$ Barbara's score.

A clean parity-colouring argument; no game-theoretic depth required, only the observation that each move alternates the parity of the indices remaining at the ends.

Question 1.277. Interviewer. Evaluate $\int \sin^n x\,dx$, $\int \cos^n x\,dx$, $\int \frac{dx}{1-\sin x}$ and $\int \frac{x^2 \sin x}{3+\sin^2 x}\,dx$.

SHELDON. (**a**) Integration by parts on $\sin^{n-1} x \cdot \sin x$ gives the standard reduction

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx,$$

terminating at $\int \sin x\,dx = -\cos x$ for n odd or $\int dx = x$ for n even.

(**b**) By the same method, or by $x \mapsto \pi/2 - x$,

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

(**c**) Multiply numerator and denominator by $1 + \sin x$:

$$\begin{aligned}\int \frac{dx}{1-\sin x} &= \int \frac{1+\sin x}{\cos^2 x}\,dx = \int (\sec^2 x + \sec x \tan x)\,dx \\ &= \tan x + \sec x + C.\end{aligned}$$

(**d**) Almost certainly the intended reading is a definite integral over symmetric limits, and there the answer is immediate: x^2 is even while $\sin x/(3+\sin^2 x)$ is odd, so the integrand is odd, and $\int_{-a}^{a} = 0$ for every a. As a bare *indefinite* integral it has no elementary closed form. Integration by parts with $u = x^2$, $dv = \sin x/(3+\sin^2 x)\,dx$, using $t = \cos x$ in dv (so $\int dv = \frac{1}{4}\ln|(2-\cos x)/(2+\cos x)|$, since $3+\sin^2 x = 4-\cos^2 x$), reduces it to

$$\int \frac{x^2 \sin x}{3+\sin^2 x}\,dx = \frac{x^2}{4}\ln\left|\frac{2-\cos x}{2+\cos x}\right| - \frac{1}{2}\int x \ln\left|\frac{2-\cos x}{2+\cos x}\right| dx,$$

whose residual integral is non-elementary. So one reports the definite-integral value 0 and stops; chasing an elementary antiderivative is a fool's errand here.

QUESTION 1.278. INTERVIEWER. Find the general solutions of (a) $y''' - 3y'' + 4y = 0$, (b) $y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$, (c) $(4x+1)y'' + 2xy' - y = (x+1)^2$.

SHELDON. (**a**) Characteristic equation $r^3 - 3r^2 + 4 = 0$. Inspection gives $r = -1$ as a root; division yields $(r+1)(r^2-4r+4) =$

$(r+1)(r-2)^2$. Hence

$$y(x) = C_1 e^{-x} + (C_2 + C_3 x)e^{2x}.$$

(b) The coefficients $1, 4, 6, 4, 1$ are binomial; the characteristic polynomial is $(r+1)^4 = 0$. Therefore

$$y(x) = (C_1 + C_2 x + C_3 x^2 + C_4 x^3)e^{-x}.$$

(c) Try a polynomial particular solution $y_p = ax^2 + bx + c$. Substituting yields $3ax^2 + (8a+b)x + (2a-c) = x^2 + 2x + 1$, hence $a = \frac{1}{3}$, $b = -\frac{2}{3}$, $c = -\frac{1}{3}$:

$$y_p = \tfrac{1}{3}(x^2 - 2x - 1).$$

The homogeneous equation $(4x+1)y'' + 2xy' - y = 0$ has a regular singular point at $x = -\frac{1}{4}$ and an ordinary point at the origin. Power-series solutions $y = \sum a_k x^k$ obey

$$(k+2)(k+1)a_{k+2} + 4k(k+1)\,a_{k+1} + (2k-1)\,a_k = 0,$$

generating two linearly independent series y_1, y_2 with no closed form in elementary functions. The general solution is $y = y_p + C_1 y_1(x) + C_2 y_2(x)$.

The first two cases yielded to the characteristic-polynomial method that Euler invented in 1739; the third was clearly inserted to test whether the candidate panics when the closed form vanishes.

Question 1.279. Interviewer. Two equal masses m on a smooth table are joined to each other and to fixed points P, Q by three identical springs (natural length a, modulus λ). With $|PQ| = 3a$, both particles are held at $XP = YQ = a/2$ and then released. Find their positions at time t.

Sheldon. Measure $x_1(t), x_2(t)$ from P. Spring PX has extension $x_1 - a$; spring XY has extension $(x_2 - x_1) - a$; spring YQ

has extension $(3a - x_2) - a$. Hooke's law with modulus λ over natural length a gives forces proportional to λ/a:

$$m\ddot{x}_1 = \tfrac{\lambda}{a}(x_2 - 2x_1), \qquad m\ddot{x}_2 = \tfrac{\lambda}{a}(x_1 - 2x_2 + 3a).$$

Equilibrium $\ddot{x}_i = 0$ gives $x_1 = a$, $x_2 = 2a$ (each spring at natural length).

Substitute $u = x_1 - a$, $v = x_2 - 2a$ and let $\omega = \sqrt{\lambda/(ma)}$. The system decouples in $u + v$ and $u - v$:

$$(u \ddot{+} v) = -\omega^2(u+v), \qquad (u \ddot{-} v) = -3\omega^2(u-v).$$

Initial data $u(0) = -a/2$, $v(0) = a/2$, $\dot{u}(0) = \dot{v}(0) = 0$ give $u + v \equiv 0$ and $u - v = -a\cos(\sqrt{3}\,\omega t)$.

Therefore

$$x_1(t) = a - \tfrac{a}{2}\cos(\sqrt{3}\,\omega t),$$
$$x_2(t) = 2a + \tfrac{a}{2}\cos(\sqrt{3}\,\omega t),$$

with $\omega = \sqrt{\lambda/(ma)}$. Only the antisymmetric mode is excited, because the initial displacement is itself antisymmetric about the midpoint $3a/2$. The symmetric (centre-of-mass) mode rests untouched throughout.

Question 1.280. Interviewer. Five children Ammar, Bilal, Chris, Dan, Egan race to a tree and back. Given: (i) Ammar fourth to the tree; (ii) last to tree won the return; (iii) first to tree was third back; (iv) third to tree was second back; (v) Bilal fourth back; (vi) Chris before Dan to tree; (vii) Chris before Egan back. Find both finishing orders.

Sheldon. Conditions (ii), (iii), (iv) link tree positions $1, 3, 5$ to back positions $3, 2, 1$ respectively. Tree position 4 is Ammar; the back positions left unassigned are 4 and 5. Since Bilal is back 4 (condition v), Bilal cannot have been at tree positions $1, 3, 4, 5$, hence Bilal is tree position 2, and Ammar (tree 4) is back 5.

The remaining children Chris, Dan, Egan fill tree positions $1, 3, 5$. Condition (vi) demands Chris before Dan; condition (vii) demands Chris back before Egan back. Map tree 1 $\to$ back 3, tree 3 $\to$ back 2, tree 5 $\to$ back 1, and check the three valid tree assignments:

- $C = 1, D = 3, E = 5$: Chris back 3, Egan back 1. Egan finishes before Chris, violates (vii).
- $C = 1, D = 5, E = 3$: Chris back 3, Egan back 2. Egan before Chris, violates (vii).
- $C = 3, D = 5, E = 1$: Chris back 2, Egan back 3. Chris before Egan, satisfies (vii).

To the tree: Egan, Bilal, Chris, Ammar, Dan.

Back from the tree: Dan, Chris, Egan, Bilal, Ammar.

The puzzle reduces to a 3×3 permutation check after the deductions from (i)–(v) trim the search space, which is the sort of analysis an admissions tutor uses to test whether the candidate can keep more than four objects straight in their head simultaneously.

Question 1.281. Interviewer. For which $\theta \in [0, 2\pi]$ is $\sin\theta + \cos\theta \geq 1$?

Sheldon. Write $\sin\theta + \cos\theta = \sqrt{2}\sin(\theta + \frac{\pi}{4})$. The inequality becomes $\sin(\theta + \pi/4) \geq 1/\sqrt{2}$, i.e.

$$\theta + \tfrac{\pi}{4} \in [\tfrac{\pi}{4} + 2k\pi, \ \tfrac{3\pi}{4} + 2k\pi].$$

Subtracting $\pi/4$ and intersecting with $[0, 2\pi]$ yields $\theta \in [0, \pi/2] \cup \{2\pi\}$.

The solution set is $\theta \in [0, \pi/2]$ together with the endpoint $\theta = 2\pi$.

The harmonic combination $a\sin\theta + b\cos\theta = \sqrt{a^2 + b^2}\sin(\theta + \varphi)$ is, incidentally, the one-line lemma that removes threequarters of the trigonometric labour in undergraduate mechanics.

Question 1.282. Interviewer. A paper disc of radius r has a sector of angle θ removed; the remainder is folded into a cone. Find the height.

Sheldon. The remaining arc has length $(2\pi - \theta)r$, which becomes the circumference of the cone's circular base, so the base radius is

$$\rho = \frac{(2\pi - \theta)\, r}{2\pi}.$$

The slant height is r (radii of the disc are unchanged by folding). By Pythagoras on the vertical cross-section,

$$h = \sqrt{r^2 - \rho^2} = \frac{r}{2\pi}\sqrt{(2\pi)^2 - (2\pi - \theta)^2} = \frac{r}{2\pi}\sqrt{\theta(4\pi - \theta)}.$$

$$h = \frac{r\sqrt{\theta(4\pi - \theta)}}{2\pi}.$$

Question 1.283. Interviewer. Show that a real number has a repeating decimal expansion iff it is rational.

Sheldon. **Rational ⇒ repeating.** Given p/q, run long division. Each step carries a remainder in $\{0, 1, \dots, q-1\}$. After at most q steps two remainders coincide; from that moment on the digit sequence cycles with period dividing the gap.

Repeating ⇒ rational. Suppose $x = A.b_1 b_2 \cdots b_n \overline{R}$ with $R = r_1 r_2 \cdots r_k$ a block of length k. Let $y = 10^n x - (Ab_1 \cdots b_n)$, so $y = 0.\overline{R}$. Then $10^k y = r_1 r_2 \cdots r_k.\overline{R} = y + R$, hence $y = R/(10^k - 1) \in \mathbb{Q}$. Therefore $x = (y + Ab_1 \cdots b_n)/10^n \in \mathbb{Q}$.

Question 1.284. Interviewer. One hundred prisoners will be visited, one per day at random, to a room with a single light. A prisoner may freely flip the switch. On any visit the prisoner may declare "all 100 of us have been here." A wrong declaration kills everyone; a correct one frees them. Before imprisonment they may agree on a strategy; no other communication is allowed. Find a strategy that guarantees release.

Sheldon. Appoint one prisoner the *counter*. Rules:

- **Any non-counter**, on entering, if the light is *off* and they have not yet turned it on, turns it on; then never touches it again on any subsequent visit.
- **The counter**, on entering, if the light is *on*, turns it off and increments a mental tally. When the tally reaches 99, declare "all have been here."

Each non-counter turns the light on exactly once in their lifetime, the first time they personally find it off. The counter clears the light each time it is on, recording the event. After 99 clearings the counter has witnessed 99 distinct non-counters having visited, plus themselves, yielding 100 unique prisoners. The declaration is then correct.

That argument assumes the light starts *off*. If the initial state is unknown, a spurious initial "on" inflates the tally by one, and simply declaring at 100 instead of 99 does not rescue it: should the light in fact start off, the tally can never exceed 99, and the counter waits forever. The honest fix doubles the signalling. **Each non-counter turns the light on the first *two* times they personally find it off** (at most twice in a lifetime, then never again); **the counter declares at a tally of** $2(100 - 1) = 198$**.**

This works whatever the initial state. Each of the 99 non-counters contributes at most 2 to the tally, and a possible initial "on" contributes at most 1 more. Suppose the tally reaches 198. If the light started off, the 198 comes entirely from non-counters, each ≤ 2, so all 99 contributed 2, hence all were visited. If it started on, the non-counters supplied at least 197; spreading 197 across 99 people with each ≤ 2 forces every one of them to contribute at least once (since $98 \times 2 = 196 < 197$), so again all 99 were visited. Either way the declaration is correct, and since in the worst case (light off) every non-counter does signal twice, the tally 198 is actually reached. The expected time to release, by coupon-collector-style analysis, is several years, which is evidently why nobody has tried the strategy empirically.

Question 1.285. Interviewer. Label a blank die with integers from $\{0, 1, \ldots, 6\}$ so that rolling it with a standard 1–6 die gives every total from 1 to 12 with equal probability.

Sheldon. Let the blank die have n_j faces showing j, so $\sum_{j=0}^{6} n_j = 6$. Each total $s \in \{1, \ldots, 12\}$ must arise with probability $1/12$, i.e. in $36/12 = 3$ outcomes. For $s = 1$ the only pair is $(1, 0)$, forcing $n_0 = 3$. For $s = 2$ pairs are $(1, 1)$ and $(2, 0)$, giving $n_1 + n_0 = 3$, so $n_1 = 0$. Inductively $n_0 = 3$ and $n_1 = n_2 = n_3 = n_4 = n_5 = 0$, then $n_6 = 3$ from the $s = 7$ count.

The blank die has *three faces marked* 0 *and three faces marked* 6.

Equivalently: the generating function $(1+x+\cdots+x^5)(\cdot)$ for the standard die times the blank die's generating function should factor $x + x^2 + \cdots + x^{12}$ up to the scaling 6. The unique-up-to-order factorisation with non-negative coefficients summing to 6 on each factor is precisely $3(1+x^6)\cdot x(1+x+x^2+x^3+x^4+x^5)$.

Question 1.286. Interviewer. If you take a Rubik's cube and repeatedly apply a fixed sequence of moves, does the cube eventually return to its initial state?

Sheldon. The configurations of the cube form a finite group G (order 43,252,003,274,489,856,000, a figure every self-respecting combinatorialist has memorised). A fixed move-sequence corresponds to an element $\sigma \in G$. Lagrange's theorem gives $\sigma^{|G|} = e$, so after at most $|G|$ repetitions the cube returns to its starting configuration. A fortiori it returns after $\operatorname{ord}(\sigma)$ applications; the longest orbit any single move sequence can have is 1260, attained by a six-move sequence discovered by Dan Hoey.

In brief: *yes*, because every element of a finite group has finite order.

Question 1.287. Interviewer. Four ice-cream sizes are revealed to you one at a time in a random order, and you must decide immediately on each. What strategy maximises the probability of choosing the largest?

Sheldon. The four-element secretary problem. Use a threshold rule: observe the first k sizes without committing, then accept the first subsequent size that exceeds all observations so far.

For n candidates the win probability is $(k/n)\sum_{i=k+1}^{n} 1/(i-1)$. With $n = 4$:

$$\begin{aligned} k = 1: &\quad \tfrac{1}{4}(1 + \tfrac{1}{2} + \tfrac{1}{3}) = \tfrac{11}{24}, \\ k = 2: &\quad \tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{3}) = \tfrac{5}{12}, \\ k = 3: &\quad \tfrac{3}{4} \cdot \tfrac{1}{3} = \tfrac{1}{4}. \end{aligned}$$

The maximum is at $k = 1$:

$$P_{\max} = \tfrac{11}{24} \approx 0.458.$$

Skip the first size, then take the first subsequent size exceeding it. As $n \to \infty$ the optimal threshold tends to n/e and the win probability to $1/e$, a piece of folklore due to Lindley and Dynkin, not to anyone called the Secretary.

Question 1.288. Interviewer. Twelve gnomes; one has anomalous weight. Three uses of a balance. Find the odd one and say whether heavier or lighter. Then: fourteen gnomes with ten known-good teddy bears available, still three weighings.

SHELDON. **Twelve gnomes.** Information-theoretic floor: three weighings give at most $3^3 = 27$ outcomes; 24 cases (12 candidates × 2 directions) fit inside.

Label the gnomes 1–12. **Weighing 1:** $\{1, 2, 3, 4\}$ vs. $\{5, 6, 7, 8\}$.

Balanced. The anomaly lies in $\{9, 10, 11, 12\}$. **Weighing 2:** $\{9, 10, 11\}$ vs. $\{1, 2, 3\}$ (known-good). If balanced, the anomaly is 12; weigh 12 against 1 to determine direction. Otherwise the direction of the tip identifies whether the anomaly in $\{9, 10, 11\}$ is heavy or light. **Weighing 3:** 9 vs. 10; balanced implies 11.

Left heavy in weighing 1. Each of $1, 2, 3, 4$ is a heavy suspect; each of $5, 6, 7, 8$ a light suspect. **Weighing 2:** $\{1, 2, 5\}$ vs. $\{3, 4, 6\}$. Balanced means the anomaly is in $\{7, 8\}$ (light); weighing 3 of 7 vs. 1 (good) settles it. Tilt left heavy implicates $1, 2$ heavy or 6 light; tilt left light implicates $3, 4$ heavy or 5 light. Three candidates remain, settled by weighing 3 (1 vs. 2, or 3 vs. 4, with an already-identified known-good as needed).

Fourteen gnomes with ten known-good. Here the interviewer has set an impossible task, and I shall say so before wasting chalk. Three weighings yield a sequence of three results, each *left-heavy*, *right-heavy*, or *balanced*: $3^3 = 27$ distinguishable outcomes in all. What must be determined is which of the fourteen gnomes is anomalous *and* whether it is heavy or light, which is $14 \times 2 = 28$ possibilities. Twenty-eight pigeons do not fit in twenty-seven holes: no procedure, with any number of known-good teddy bears, can map 28 cases injectively onto 27 outcomes. The bears supply references, not outcomes, so they cannot rescue it.

The honest ceiling is thirteen. With at least one known-good reference in hand, a w-weighing scheme distinguishes $(3^w - 1)/2$ cases-with-direction; for $w = 3$ that is 13, and 14 exceeds it. I would suggest the interviewer either retire one gnome from suspicion or grant me a fourth weighing. A theoretical physicist does not pretend a measurement carries more information than it does; that is what experimentalists are for.

The twelve-coin problem was first posed by Dyson in 1946 to Schrödinger in the Bohr Institute tea-room, or so Dyson liked to claim.

Question 1.289. Interviewer. Ten bottles of 2000 pills, paracetamol at 1000 mg each. (i) One has been replaced by ibuprofen, 1010 mg each. With a single placement on an electronic scale, identify which. (ii) Now any subset of bottles has been replaced; identify every ibuprofen bottle with a single weighing.

Sheldon. **(i)** Take k pills from bottle k for $k = 1, \dots, 10$: total 55 pills. All-paracetamol mass is $55 \cdot 1000 = 55{,}000$ mg; bottle k ibuprofen adds $10k$ mg to the reading. The bottle number is

$$k = \frac{\text{mass} - 55{,}000}{10}.$$

(ii) Take 2^{k-1} pills from bottle k: totals $1 + 2 + \cdots + 512 = 1023$ pills. All-paracetamol mass is 1,023,000 mg; each ibuprofen bottle k contributes an extra $10 \cdot 2^{k-1}$ mg. Let M be the excess over 1,023,000 mg. Then $M/10$ is the sum of the 2^{k-1} corresponding to ibuprofen bottles; its binary expansion reads off the set of bottles directly.

For (i), $k = (M - 55{,}000)/10$; for (ii), the excess mass over 1,023,000 mg divided by 10 is the binary indicator of ibuprofen bottles.

Question 1.290. Interviewer. An equilateral triangle of side N is partitioned into N^2 unit-triangle cells. A route starts in the corner cell at vertex A, steps only through shared edges, and visits no cell twice. Find, with proof, the maximum number of cells visited.

Sheldon. Colour each cell by orientation: $\triangle$ (apex-up) or $\bigtriangledown$ (apex-down). Any shared edge is between one up and one down cell, so the adjacency graph is bipartite. Counting: the k-th row from the top contains k up-cells and $k-1$ down-cells, giving

$$|\triangle| = \tfrac{N(N+1)}{2}, \qquad |\bigtriangledown| = \tfrac{N(N-1)}{2}, \qquad |\triangle| - |\bigtriangledown| = N.$$

Any simple path alternates colours, hence the difference between the number of cells of the two colours it uses is 0 or 1. Therefore it visits at most $|\bigtriangledown| + (|\bigtriangledown| + 1) = 2 \cdot \frac{N(N-1)}{2} + 1 = N^2 - N + 1$ cells.

The bound is attained by a serpentine path: traverse row 1 (the corner cell), drop into row 2, snake across, drop into row 3, and so on. Such a path uses every one of the $\frac{N(N-1)}{2}$ down-cells and exactly $\frac{N(N-1)}{2} + 1$ up-cells, that is, just *one* more up-cell than down-cell in total (the single surplus of the starting colour), for $N^2 - N + 1$ cells altogether. It leaves $N - 1$ up-cells unvisited.

The maximum is $N^2 - N + 1$ cells.

Question 1.291. Interviewer. Evaluate $\int_0^{\pi/2} \dfrac{\sin^2 x}{2 + \sin x}\, dx$, $\displaystyle\int \frac{dx}{(1+x^2)^2}$, $\displaystyle\int \frac{dx}{1+x^4}$, $\displaystyle\int \frac{1+x^5}{(1+x^2)^3}\, dx$.

Sheldon. **(a)** Long-divide: $\sin^2 x = (\sin x + 2)(\sin x - 2) + 4$, so $\sin^2 x/(2 + \sin x) = \sin x - 2 + 4/(2 + \sin x)$. The first part integrates to $1 - \pi$ on $[0, \pi/2]$. For the remainder, Weierstrass $t = \tan(x/2)$ converts $\int_0^{\pi/2} 4/(2 + \sin x)\, dx$ to $\int_0^1 4/(t^2 + t + 1)\, dt = \frac{4\pi}{3\sqrt{3}} = \frac{4\pi\sqrt{3}}{9}$. Total:

$$1 - \pi + \tfrac{4\pi\sqrt{3}}{9}.$$

(b) Set $x = \tan\theta$:

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2\theta\, d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \tfrac{1}{2}\arctan x + \tfrac{x}{2(1+x^2)} + C.$$

(c) Factor $1 + x^4 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$ and split into partial fractions. After grouping,

$$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \ln\left|\frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1}\right| + \frac{1}{2\sqrt{2}} \arctan\left(\frac{x^2 - 1}{x\sqrt{2}}\right) + C.$$

(d) Split $\int (1+x^5)/(1+x^2)^3 = \int (1+x^2)^{-3} + \int x^5(1+x^2)^{-3}$. For the second, $u = 1 + x^2$: $\int (u-1)^2/(2u^3)\,du = \tfrac{1}{2}\ln(1+x^2) + \tfrac{1}{1+x^2} - \tfrac{1}{4(1+x^2)^2}$ plus constant. For the first, $x = \tan\theta$ gives $\int \cos^4\theta\,d\theta = \tfrac{3\theta}{8} + \tfrac{\sin 2\theta}{4} + \tfrac{\sin 4\theta}{32}$, which rewrites as $\tfrac{3}{8}\arctan x + \tfrac{x}{2(1+x^2)} + \tfrac{x(1-x^2)}{8(1+x^2)^2}$. Combining:

$$\int \frac{1+x^5}{(1+x^2)^3}\,dx = \tfrac{1}{2}\ln(1+x^2) + \tfrac{3}{8}\arctan x + \tfrac{x}{2(1+x^2)} + \tfrac{1}{1+x^2} + \tfrac{x(1-x^2)}{8(1+x^2)^2} - \tfrac{1}{4(1+x^2)^2} + C.$$

Question 1.292. Interviewer. Solve the system $\dfrac{dy}{dt} = -2(y-z)$, $\dfrac{dz}{dt} = \dfrac{dy}{dt} - 3z$.

Sheldon. Substitute the first equation into the second: $\dot{z} = -2(y-z) - 3z = -2y - z$. The system is

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Characteristic polynomial $(\lambda + 2)(\lambda + 1) + 4 = \lambda^2 + 3\lambda + 6$, eigenvalues $\lambda = (-3 \pm i\sqrt{15})/2$. Hence

$$y(t) = e^{-3t/2}(C_1 \cos \tfrac{\sqrt{15}\,t}{2} + C_2 \sin \tfrac{\sqrt{15}\,t}{2}),$$

and $z(t)$ follows from $2z = 2y + \dot{y}$:

$$z(t) = e^{-3t/2}[(\tfrac{C_1}{4} + \tfrac{\sqrt{15}}{4}C_2)\cos \tfrac{\sqrt{15}\,t}{2} + (\tfrac{C_2}{4} - \tfrac{\sqrt{15}}{4}C_1)\sin \tfrac{\sqrt{15}\,t}{2}].$$

Two arbitrary constants, as one expects for a second-order real system. Both oscillations decay with envelope $e^{-3t/2}$, ensuring the origin is a stable spiral.

Question 1.293. Interviewer. A function f satisfies $f''(x) = -f(x)$, $f(0) = 0$, $f'(0) = 1$. Prove (a) $f(x+a) = f(x)f'(a) + f'(x)f(a)$; (b) $f(x)^2 + f'(x)^2 = 1$; (c) there exists x with $f(x+a) = f'(x)$.

Sheldon. **(a)** Fix a. Set $g(x) = f(x+a)$ and $h(x) = f(x)f'(a) + f'(x)f(a)$. Both satisfy $y'' = -y$ (differentiate h twice using $f''' = -f', f'' = -f$), and at $x = 0$:

$$g(0) = h(0) = f(a), \qquad g'(0) = h'(0) = f'(a),$$

using $f''(0) = -f(0) = 0$. By the uniqueness of solutions to the stated IVP, $g = h$ identically.

(b) Let $E(x) = f(x)^2 + f'(x)^2$. Then $E'(x) = 2ff' + 2f'f'' = 2f'(f + f'') = 0$ since $f'' = -f$. Hence E is constant, and $E(0) = 0^2 + 1^2 = 1$.

(c) Read the claim with the same a as in part (a) fixed but *arbitrary*: I must produce an x with $f(x+a) = f'(x)$. Substituting the addition formula from (a),

$$f(x+a)-f'(x) \;=\; f(x)f'(a)+f'(x)f(a)-f'(x) \;=\; f'(a)f(x)-(1-f(a))f'(x),$$

so it suffices to find x with

$$f'(a)f(x) \;=\; (1-f(a))f'(x). \qquad (*)$$

Consider $g(x) = f(x+a) - f'(x)$, a continuous function. It is 2π-periodic (both f and f' are, once one knows the IVP pins f down; even without naming f, g has the period of the system). Over one period $\int_0^{2\pi} g = \int_0^{2\pi} f(x+a)\,dx - \int_0^{2\pi} f'(x)\,dx = 0 - [f(2\pi) - f(0)] = 0$, since f shares $f(0)$'s value after a full period. A continuous periodic function of mean zero is either identically zero or takes both signs; by the intermediate value theorem it therefore has a zero. Hence some x satisfies $g(x) = 0$, which is exactly $(*)$.

The degenerate case is instructive. If $a \equiv \pi/2 \pmod{2\pi}$ then $f(a) = 1$ and $f'(a) = 0$, $(*)$ collapses to $0 = 0$, and *every* x works. If $a \equiv -\pi/2$ then $f(a) = -1, f'(a) = 0$, and $(*)$ reads $0 = 2f'(x)$, solved wherever $f'(x) = 0$. For all other a, $(*)$ is $pf(x) = qf'(x)$ with $(p, q) \neq (0, 0)$, i.e. $p \sin x = q \cos x$, solved by $x = \arctan(q/p)$. So a solution exists for every a. Recognising f as sin is unnecessary; the IVP plus (a) suffices.

QUESTION 1.294. INTERVIEWER. Find all integers coprime to every term of the sequence $u_n = 2^n + 3^n + 6^n - 1$.

SHELDON. Claim: ± 1 only. Equivalently, every prime p divides some u_n.

$p = 2$. $u_n = \underbrace{2^n}_{\text{even}} + \underbrace{3^n + 6^n - 1}_{\text{odd+even−odd=even}}$, so $2 \mid u_n$ for all n.

$p = 3$. $u_n \equiv 2^n - 1 \pmod 3$, and $2^n \equiv 1 \pmod 3$ whenever n is even. So $3 \mid u_2$.

$p \geq 5$. By Fermat, $2^{p-2} \equiv 2^{-1}, 3^{p-2} \equiv 3^{-1}, 6^{p-2} \equiv 6^{-1}$ modulo p. Therefore

$$2^{p-2} + 3^{p-2} + 6^{p-2} \equiv \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{3 + 2 + 1}{6} = 1 \pmod p.$$

Hence $u_{p-2} \equiv 0 \pmod p$.

Every prime divides some term, so the only integers coprime to all u_n are ± 1.

± 1.

The identity $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$ is the same one the Egyptians used to express fractions as sums of unit fractions in the Rhind papyrus, some four thousand years before Fermat.

QUESTION 1.295. INTERVIEWER. A social network has 2019 users. When A is friends with both B and C but B, C are not friends, a move deletes edges AB, AC and adds edge BC. Initially 1010 users have 1009 friends each and 1009 users have 1010 friends each. Prove a sequence of moves leaves every user with at most one friend.

SHELDON. **Three invariants.** Look at what one move, with apex A on the pair B, C, does to the degrees. A loses the two edges AB, AC, so $\deg A$ drops by 2. B loses AB but gains BC, so $\deg B$ is *unchanged*; likewise $\deg C$. Hence:

- the edge count $|E|$ falls by exactly 1 per move;
- the *parity* of every single vertex's degree is preserved forever (only the apex changes degree, and by 2).

Initially $|E_0| = \frac{1}{2}(1010 \cdot 1009 + 1009 \cdot 1010) = 1009 \cdot 1010$, and there are exactly 1010 vertices of odd degree (1009) and 1009 of even degree (1010). A matching (every degree ≤ 1) has its odd-degree vertices matched in pairs; since the number of odd-degree vertices is the invariant 1010, any matching we reach must use exactly $1010/2 = 505$ edges and leave 1009 vertices isolated. The parities line up, so a matching is not excluded on counting grounds; the work is to *reach* one.

The naive algorithm fails. It is tempting to say "keep moving while a move exists." But a move requires a vertex with two *non*-adjacent neighbours, and a configuration admits no move precisely when every vertex's neighbourhood is a clique, which forces every connected component to be a complete graph. (If $u \sim v \sim w$ then u, w are both neighbours of v, so $u \sim w$; adjacency is transitive on a component, making it complete.) A lone triangle K_3 is such a dead end, and it is *not* a matching. So a careless sequence can strand us at, say, 505 disjoint edges plus one isolated triangle plus 1006 isolated vertices, which the parity count above permits. A correct argument must *choose* its moves to avoid ever isolating a clique of three or more.

A strategy that does reach a matching. Two facts do the work.

1. *Forests always finish.* In a forest, any two neighbours of a vertex are non-adjacent (an edge between them would close a cycle), so *every* vertex of degree ≥ 2 offers a legal move, and the move keeps the graph a forest with one fewer edge. Edge count cannot fall forever, so a forest is driven all the way down to maximum degree ≤ 1, a matching.

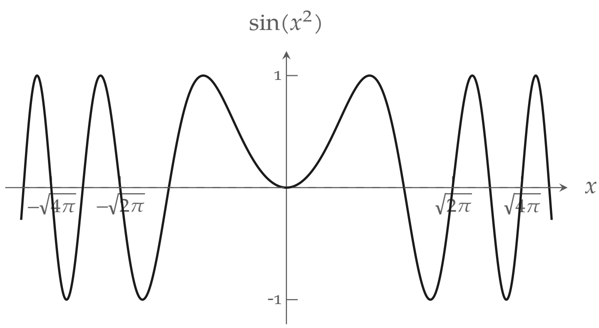
2. *Cycles can be removed without stranding a triangle.* While a cycle remains, take a shortest cycle and an apex A on it whose two cycle-neighbours B, C are non-adjacent (in a shortest cycle of length ≥ 4 they are); the move deletes two cycle edges and adds the chord BC, shortening that cycle by one. Repeating turns long cycles into triangles. A triangle is broken only with help from outside: if one of its vertices A also has an external neighbour D, then D is non-adjacent to a triangle-mate B, and the move on $A; B, D$ destroys the edge AB and opens the triangle. The only triangle one cannot break is an *isolated* one, so the rule is: never isolate a triangle. Because the final matching pairs an even number (1010) of vertices, the endgame never *forces* a leftover odd clique; whenever a component is about to collapse to an isolated triangle, the surplus odd-degree vertices elsewhere let us instead route an edge out and keep it attached until it can be opened.

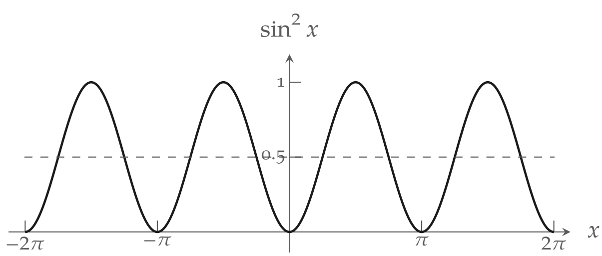
Apply (2) until the graph is a forest, then (1) to finish. Every degree ends at most 1, as required.

This is the kind of social-network problem the olympiad circuit fielded around 2019; it separated the candidates who knew the word "monovariant" from those who merely decremented an edge count and hoped.

Question 1.296. Interviewer. Sketch $y = \sin(x^2)$ and $y = \sin^2 x$. What shape does the first take near the origin?

SHELDON. The two are easily confused and behave quite differently. The square $\sin^2 x$ is genuinely non-negative, periodic with period π, oscillating in $[0, 1]$ with bumps centred at half-integer multiples of π. By contrast $\sin(x^2)$ is *not* non-negative: it dips to -1 wherever $x^2 \equiv 3\pi/2 \pmod{2\pi}$, the first such trough at $x = \sqrt{3\pi/2} \approx 2.17$, so it swings through the full range $[-1, 1]$ like an ordinary sine, only with an argument that accelerates. It loses periodicity for that reason. Near the origin, $\sin(x^2) \approx x^2$: a parabola kissing the x-axis from above. Further out the oscillations of $\sin(x^2)$ compress rapidly because consecutive zeros $\sqrt{k\pi}$ cluster as k grows.





The parabolic onset of $\sin(x^2)$ near the origin is exactly what one observes in the Fresnel integral, which is essentially $\int_0^x \sin(t^2)\,dt$ and gives rise to the Cornu spiral in diffraction optics.

QUESTION 1.297. INTERVIEWER. An inverted cone of height h is filled halfway up. What proportion of its volume is empty?

Sheldon. The water forms a smaller cone similar to the whole, with linear scale $1/2$. Volumes scale as the cube of the linear factor, so the water occupies $(1/2)^3 = 1/8$ of the total volume. The empty portion is therefore

$$1 - \tfrac{1}{8} = \tfrac{7}{8}.$$

The empty fraction is $7/8$.

The "halfway-up holds only an eighth" observation trips up every first-year engineer designing a conical funnel, which is why industrial chemistry leans on cylinders instead.

Question 1.298. Interviewer. Use the geometric-series formula to prove that $0.99999\ldots = 1$.

Sheldon. The decimal expansion is the series

$$0.999\ldots = \sum_{n=1}^{\infty} \frac{9}{10^n} = 9 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n.$$

The geometric series with ratio $1/10$ and first term $1/10$ sums to $\frac{1/10}{1-1/10} = \frac{1}{9}$, so the total is $9 \cdot \frac{1}{9} = 1$.

This is not a limit "almost equal to" 1. It is equal, because the decimal is a definition of a real number and the series converges absolutely to that number. Any remaining discomfort is psychological, not mathematical.

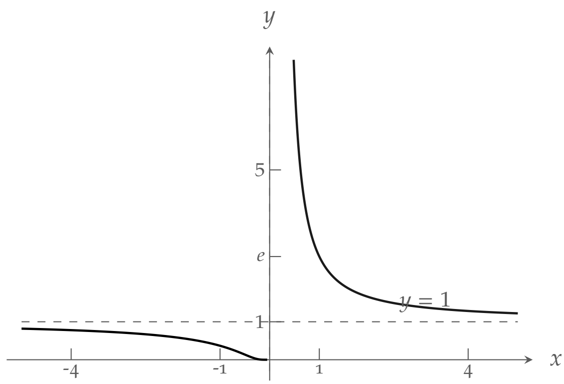
Question 1.299. Interviewer. Sketch $y = e^{1/x}$.

Sheldon. Domain $x \neq 0$. Behaviour at the edges:

- $x \to +\infty$: $1/x \to 0^+$, $y \to 1^+$ (horizontal asymptote $y = 1$ from above).
- $x \to 0^+$: $1/x \to +\infty$, $y \to +\infty$ (vertical asymptote from the right).
- $x \to 0^-$: $1/x \to -\infty$, $y \to 0^+$ (the curve hugs the x-axis, approaching but not touching zero).

- $x \to -\infty$: $1/x \to 0^-$, $y \to 1^-$ (asymptote from below).

Derivative: $y' = -y/x^2 < 0$ for all $x \neq 0$, since $y > 0$ and $x^2 > 0$. So y is strictly decreasing on each of $(-\infty, 0)$ and $(0, \infty)$, reading in the direction of increasing x. On the left branch this means y falls from 1^- (as $x \to -\infty$) down to 0^+ (as $x \to 0^-$); on the right branch it falls from $+\infty$ (as $x \to 0^+$) down to 1^+ (as $x \to +\infty$).



The dichotomy at the origin is a standard test of whether a candidate properly distinguishes one-sided limits, as opposed to lazily writing $e^{1/0}$ and hoping for the best.

Question 1.300. Interviewer. Given $a \neq b$: how many real roots does $(x-a)^2 + (x-b)^2 = 0$ have? Show that $(x-a)^3 + (x-b)^3 = 0$ has exactly one real root.

Sheldon. **First equation.** A sum of real squares vanishes only when each summand does, demanding $x = a$ and $x = b$ simultaneously. Since $a \neq b$, no real x qualifies; the equation has **no real roots**. (Over $\mathbb{C}$ it has two conjugate roots $(a+b)/2 \pm i|a-b|/2$.)

Second equation. Factor the sum of cubes:

$$(x-a)^3 + (x-b)^3 = ((x-a) + (x-b))$$

$$\cdot\left((x-a)^2-(x-a)(x-b)+(x-b)^2\right).$$

The first factor vanishes iff $x = (a+b)/2$. The second factor equals $u^2 - uv + v^2$ with $u = x - a$, $v = x - b$. Completing the square:

$$u^2 - uv + v^2 \;=\; \left(u - \tfrac{v}{2}\right)^2 + \tfrac{3v^2}{4} \;\geq\; 0,$$

with equality iff $u = v = 0$, i.e. $x = a = b$, impossible by hypothesis. Hence the quadratic factor is strictly positive, and the unique real root is

$$x = \tfrac{a+b}{2}.$$