Problems in Classical and Contemporary Mathematics

A curated selection

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PREFACE

This volume gathers problems the editor has found particularly beautiful over the years, drawn both from the classical canon and from the work of contemporary composers. The selection is curatorial: each problem has been judged on the elegance of its solution, not on originality or difficulty alone, and where a source or composer is known the attribution is recorded. No claim is made to authorship of the problems themselves. The editor's contribution is the selection, the grouping by technique, the typesetting, and the occasional rewriting of solutions that could be tightened.

The chapters are organised by the tool that most naturally unlocks the problems they contain: generating functions for series, parameter differentiation for integrals, conditioning arguments for probability, and so forth. Within each chapter the problems are arranged roughly by ascending technical weight, starting from the accessible and ending in the virtuosic. A diligent reader can work through the book in order; a reader in search of a particular technique can open any chapter and find its emblematic problems at the front.

Contents

Chapter 1

Series and Summations

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THE PROBLEMS IN THIS CHAPTER CONCERN CLOSED-FORM EVALUATION OF INFINITE SERIES. Generating functions are the recurring instrument: identify an ordinary or exponential generating function whose coefficients match the summand, substitute a convenient value, and read off the answer. The Fibonacci generating function $x/(1-x-x^2)$ and the harmonic generating function $-\ln(1-x)/(1-x)$ each appear here in several disguises. Where the summand splits cleanly mod 4, the Taylor series of sin and cos at $\pm\pi$ unlock unexpected closed forms; this is the engine behind the Raghava sums. Two problems telescope, the quickest method when it applies; one swaps the order of a double sum and reduces to a Beta-function integral. Throughout, the goal is the cleanest path to a closed form, not the shortest. Several problems below were composed by K. S. Raghava and D. Rattaggi and have circulated only in problem columns; the editor records his debt to both.

Problem 1.1

Prove that

$$\sum_{n=1}^{\infty} \frac{F_n}{a^{n+1}} = \sum_{n=1}^{\infty} \frac{F_{2n}}{(a+1)^{n+1}},$$

where F_n is the n-th Fibonacci number.

D. Rattaggi

Solution. The ordinary generating function for the Fibonacci sequence is $f(x) = \sum_{n=1}^{\infty} F_n x^n = x/(1-x-x^2)$. The even-indexed subsequence has generating function $g(x^2) = (f(x)+f(-x))/2 = x^2/(1-3x^2+x^4)$, so that $g(x) = \sum_{n=1}^{\infty} F_{2n} x^n = x/(1-3x+x^2)$.

Setting $x = 1/a$ in f:

$$\frac{1}{a} f\left(\tfrac{1}{a}\right) = \sum_{n=1}^{\infty} \frac{F_n}{a^{n+1}} = \frac{1}{a^2-a-1}.$$

Setting $x = 1/(a+1)$ in g:

$$\frac{1}{a+1} g\left(\tfrac{1}{a+1}\right) = \sum_{n=1}^{\infty} \frac{F_{2n}}{(a+1)^{n+1}} = \frac{1}{a^2-a-1}.$$

The two sums agree, as claimed.

Problem 1.2

Evaluate

$$\sum_{n=0}^{\infty} \frac{(-1)^{\lceil n/2 \rceil} - (-1)^n}{n!}.$$

D. Rattaggi

Solution. The parity table of $(-1)^{\lceil n/2 \rceil} - (-1)^n$ has period 4: it takes the values $0, 0, -2, 2$ for $n \equiv 0, 1, 2, 3 \pmod 4$. Therefore

$$\begin{aligned} S = {} & -2 \sum_{n=0}^{\infty} \frac{1}{(4n+2)!} \\ & + 2 \sum_{n=0}^{\infty} \frac{1}{(4n+3)!}. \end{aligned}$$

The quarter-period identities $\sum_n 1/(4n+2)! = \cosh(1) - \cos(1)$ and $\sum_n 1/(4n+3)! = \sinh(1) - \sin(1)$ (each obtained from the four quarter-series of e^x at $x = 1$) give

$$S = \cos(1) - \cosh(1) + \sinh(1) - \sin(1),$$

a value approximately equal to -0.669.

PROBLEM 1.3

Prove that

$$\sum_{n=1}^{\infty} \frac{1+2+\cdots+n}{1\cdot 2\cdots n}(-1)^{1+2+\cdots+n}\,\pi^n \;=\; \frac{\pi(\pi+2)}{2}.$$

K. S. Raghava

Solution. The triangular-number coefficient is $T_n = n(n+1)/2$, and $(-1)^{T_n}$ has period 4 in n: it takes the values $-1, -1, +1, +1$ for $n \equiv 1, 2, 3, 0 \pmod 4$. Write

$$S(x) \;=\; \tfrac{1}{2}\sum_{n\ge 1} \frac{n(n+1)}{n!}\,(-1)^{T_n}\,x^n.$$

Shifting $n \mapsto n+1$ in $n(n+1)x^n/n! = (n+1)x \cdot x^{n-1}/(n-1)! + nx^2 \cdot x^{n-2}/(n-2)!$, the sum splits into pieces involving $\sum x^n/(n!)\,(-1)^{T_n}$ and its first derivative. The key identities, obtained by separating the four residue classes mod 4 in the Taylor series of cos and sin, are

$$\sum_{n\ge 0} \frac{x^{2n}}{(2n)!}\,(-1)^n \;=\; \cos x, \qquad \sum_{n\ge 0} \frac{x^{2n+1}}{(2n+1)!}\,(-1)^n \;=\; \sin x.$$

Setting $x = \pi$: the cosine series gives $\cos\pi = -1$ and the sine series gives $\sin\pi = 0$. The shifted sums in the expression for $S(\pi)$ collapse to $\pi^2/2$ from the second-order shift and π from the first-order shift, giving

$$S(\pi) \;=\; \frac{\pi^2}{2} + \pi \;=\; \frac{\pi(\pi+2)}{2}.$$

PROBLEM 1.4

Evaluate

$$\sum_{n=0}^{\infty} \arctan\frac{2}{(2n+1)^2}.$$

Solution. Observe the telescoping identity

$$\arctan(2n+2) - \arctan(2n) \;=\; \arctan\frac{2}{(2n+1)^2},$$

obtained by applying $\arctan a - \arctan b = \arctan\big((a-b)/(1+ab)\big)$ to the two adjacent even integers. The original sum therefore telescopes to

$$\sum_{n=0}^{\infty} \arctan\frac{2}{(2n+1)^2} \;=\; \lim_{N\to\infty} \arctan(2N) - \arctan(0) \;=\; \frac{\pi}{2}.$$

Problem 1.5

Evaluate

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Solution. Differentiating the geometric series $\sum_{n\geq 0} x^n = 1/(1-x)$ with respect to x gives $\sum_{n\geq 1} nx^{n-1} = 1/(1-x)^2$. Therefore $\sum_{n\geq 1} nx^n = x/(1-x)^2$, and setting $x = 1/2$ produces $\sum n/2^n = (1/2)/(1/4) = 2$.

Problem 1.6

Evaluate

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

Solution. Partial fractions give $\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$. The sum telescopes two-at-a-time:

$$\begin{aligned}\sum_{n=2}^{N} \tfrac{1}{n^2-1} &= \tfrac{1}{2}\sum_{n=2}^{N}\Big(\tfrac{1}{n-1} - \tfrac{1}{n+1}\Big)\\ &= \tfrac{1}{2}\Big(1 + \tfrac{1}{2} - \tfrac{1}{N} - \tfrac{1}{N+1}\Big).\end{aligned}$$

As $N \to \infty$, the sum approaches $3/4$.

PROBLEM 1.7

Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Solution. This is a classical evaluation due to Euler. The identity

$$\arcsin^2(\tfrac{x}{2}) \;=\; \tfrac{1}{2} \sum_{n\geq 1} \frac{x^{2n}}{n^2\binom{2n}{n}},$$

derived by squaring the Taylor series of arcsin and integrating, yields at $x = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} \;=\; 2\arcsin^2(\tfrac{1}{2}) \;=\; \frac{\pi^2}{18}.$$

PROBLEM 1.8

Evaluate

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the n-th harmonic number.

Solution. Start from the generating function

$$\sum_{n\geq 1} H_n x^n \;=\; -\frac{\ln(1-x)}{1-x}.$$

Dividing by x and integrating from 0 to t yields

$$\sum_{n=1}^{\infty} \frac{H_n}{n} t^n \;=\; \tfrac{1}{2}\ln^2(1-t) + \mathrm{Li}_2(t),$$

where $\mathrm{Li}_2(t) = \sum t^n/n^2$. Set $t = 1/2$ and substitute the special value $\mathrm{Li}_2(\frac{1}{2}) = \pi^2/12 - (\ln 2)^2/2$:

$$\sum_{n=1}^{\infty} \frac{H_n}{n \cdot 2^n} \;=\; \tfrac{1}{2}\ln^2 2 + \tfrac{\pi^2}{12} - \tfrac{\ln^2 2}{2} \;=\; \frac{\pi^2}{12}.$$

Problem 1.9

Evaluate

$$\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{k^n\cdot k!}.$$

K. S. Raghava

Solution. Swap the order of summation; the inner geometric series in $1/k$ gives

$$S \;=\; \sum_{k\geq 2}\frac{1}{k!}\sum_{n\geq 2}\frac{1}{k^n} \;=\; \sum_{k\geq 2}\frac{1}{k\,(k-1)\cdot k!}.$$

Use the Beta-function identity $\frac{1}{k(k-1)} = \int_0^1 (1-x)\,x^{k-2}\,dx$ and exchange sum and integral:

$$S \;=\; \int_0^1 (1-x)\sum_{k\geq 2}\frac{x^{k-2}}{k!}\,dx \;=\; \int_0^1 (1-x)\cdot\frac{e^x-1-x}{x^2}\,dx.$$

Split the integrand:

$$S \;=\; \int_0^1 \frac{e^x-1-x}{x^2}\,dx \;-\; \int_0^1 \frac{e^x-1-x}{x}\,dx.$$

Integration by parts on the first integral, with $u = e^x-1-x$ and $dv = dx/x^2$, yields $[-(e^x-1-x)/x]_0^1 + \int_0^1 (e^x-1)/x\,dx = -(e-2)+\mathrm{Ein}(1)$. The second integral equals $\mathrm{Ein}(1)-1$ (separating the linear part). Subtracting, the non-elementary $\mathrm{Ein}(1)$ terms cancel:

$$S \;=\; (2-e+\mathrm{Ein}(1)) - (\mathrm{Ein}(1)-1) \;=\; 3-e.$$

Problem 1.10

Let the sum

$$S \;=\; \sum\frac{1}{m^2\,n^2}$$

range over all pairs (m,n) of positive integers such that the largest power of 2 dividing m differs from the largest power of 2 dividing n. Express S in the form $\alpha\,\pi^k$.

R. Stanley

Solution. Write every positive integer as $2^i u$ with u odd. The constraint $i \neq j$ lets the sum factor as

$$S = \Bigg(\sum_{u \text{ odd}} \frac{1}{u^2} \Bigg)^2 \sum_{\substack{i,j \geq 0 \\ i \neq j}} \frac{1}{4^i \cdot 4^j}.$$

The odd-only Basel value is $\sum_{u \text{ odd}} u^{-2} = (1 - 1/4)\,\zeta(2) = \pi^2/8$. The double geometric sum is $(\frac{4}{3})^2 - \frac{16}{15} = 32/45$. Multiplying,

$$S = \frac{\pi^4}{64} \cdot \frac{32}{45} = \frac{\pi^4}{90}.$$

Problem 1.11

Let $p > q$ be the roots of $x^2 - x - 1 = 0$. Define $a_n = (p^n - q^n)/(p - q)$ (the Fibonacci sequence) and let $b_1 = 1$, $b_n = a_{n-1} + a_{n+1}$ for $n \geq 2$. Evaluate

$$\sum_{n=1}^{\infty} \frac{b_n}{10^n}.$$

Solution. The recurrence $a_{n-1} + a_{n+1} = L_n$ (Lucas numbers) means $b_n = L_n$ for $n \geq 2$, with $b_1 = 1$ (slightly smaller than $L_1 = 1$, so no correction needed). Using the closed form $L_n = p^n + q^n$ and summing the two geometric series,

$$\sum_{n \geq 1} \frac{p^n}{10^n} = \frac{p}{10 - p}, \qquad \sum_{n \geq 1} \frac{q^n}{10^n} = \frac{q}{10 - q}.$$

With $p + q = 1$ and $pq = -1$,

$$\sum_{n \geq 1} \frac{L_n}{10^n} = \frac{p(10 - q) + q(10 - p)}{(10 - p)(10 - q)} = \frac{10 - 2pq}{100 - 10 + pq} = \frac{12}{89}.$$

Both 12 and $89 = F_{11}$ emerge from Fibonacci arithmetic.

CHAPTER 2

Integrals

THE PROBLEMS IN THIS CHAPTER EVALUATE DEFINITE INTEGRALS IN CLOSED FORM. Three techniques recur. First, the Feynman parameter trick: introduce a parameter, differentiate under the integral sign, integrate the simpler problem, then recover the original by parameter integration. Second, a substitution that exploits a hidden symmetry; the Weierstrass half-angle substitution converts trigonometric integrals to rational ones, and the involution $x \mapsto 1/x$ folds an integral onto itself. Third, expansion as a series (Taylor or Fourier) followed by termwise integration. One problem evaluates a Riemann sum disguised as a series, and one is computed by recognising the integrand as the expectation of a function of binary digits. The aim throughout is the closed form, not a numerical approximation.

PROBLEM 2.1

Prove that

$$\int_0^{\pi/2} \frac{x}{\tan x}\, dx = \frac{\pi \ln 2}{2}.$$

Solution. Substituting $u = \tan x$ gives

$$\int_0^{\pi/2} \frac{x}{\tan x}\, dx = \int_0^{\infty} \frac{\arctan u}{u\,(u^2+1)}\, du.$$

Parameterise by

$$I(a) = \int_0^{\infty} \frac{\arctan(au)}{u\,(u^2+1)}\, du.$$

Differentiation under the integral sign, using partial fractions on $1/((u^2+1)(a^2u^2+1))$, yields

$$I'(a) \;=\; \frac{\pi}{2(1+a)},$$

so that $I(a) = \frac{1}{2}\pi\ln(1+a)$ after integrating and applying $I(0) = 0$. Setting $a = 1$ delivers $I(1) = \pi\ln 2/2$.

PROBLEM 2.2

Evaluate

$$\int_0^{\pi/2} \ln(\sin x)\,dx.$$

Solution. Call the integral I. Substituting $x \mapsto \pi/2 - x$ shows $I = \int_0^{\pi/2}\ln(\cos x)\,dx$. Therefore

$$2I \;=\; \int_0^{\pi/2} \ln(\sin x\cos x)\,dx \;=\; \int_0^{\pi/2} \ln(\tfrac{\sin 2x}{2})\,dx.$$

The change of variable $u = 2x$ on the right gives

$$2I \;=\; \frac{1}{2}\int_0^{\pi} \ln(\sin u)\,du \;-\; \frac{\pi}{2}\ln 2.$$

By symmetry the remaining integral is $2I$, so $2I = I - \frac{\pi}{2}\ln 2$, giving $I = -\frac{\pi}{2}\ln 2$.

PROBLEM 2.3

Evaluate

$$\int_{-\infty}^{\infty} \left(\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}\,F_n}{a^{n+1}} \right) da,$$

where F_n is the n-th Fibonacci number.

K. S. Raghava

Solution. Let $F(x) = \sum_{n\geq 0} F_n x^n = x/(1-x-x^2)$. The generating functions for the even-indexed and odd-indexed Fibonacci subsequences are

$$F_e(x) \;=\; \sum_{n\geq 0} F_{2n}x^n \;=\; \frac{x}{1-3x+x^2},$$

$$F_o(x) \;=\; \sum_{n\geq 0} F_{2n+1}x^n \;=\; \frac{1-x}{1-3x+x^2}.$$

The sign pattern $(-1)^{n(n-1)/2}$ has period 4 and combines these two subsequences, giving

$$\sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}F_n}{a^{n+1}} \;=\; \frac{a^2-a+1}{a^4+3a^2+1}.$$

A rational-function integration (partial fractions, logarithmic and arctangent antiderivatives) produces

$$\int_{-\infty}^{\infty} \frac{a^2-a+1}{a^4+3a^2+1}\,da \;=\; \frac{2\pi}{\sqrt{5}}.$$

Problem 2.4

Evaluate, for $a > 0$,

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x}\,dx.$$

Solution. This is a Frullani integral. Writing

$$\frac{e^{-ax}-e^{-bx}}{x} \;=\; \int_a^b e^{-tx}\,dt,$$

and swapping the order of integration,

$$\int_0^{\infty}\int_a^b e^{-tx}\,dt\,dx \;=\; \int_a^b\int_0^{\infty} e^{-tx}\,dx\,dt \;=\; \int_a^b \frac{dt}{t} \;=\; \ln\frac{b}{a}.$$

Problem 2.5

Evaluate

$$\int_0^{\infty} \frac{\sin x}{x}\,dx.$$

Solution. Parameterise: $I(a) \;=\; \int_0^{\infty} e^{-ax}\frac{\sin x}{x}\,dx$ for $a \;\geq\; 0$. Differentiating under the integral sign:

$$I'(a) \;=\; -\int_0^{\infty} e^{-ax}\sin x\,dx \;=\; -\frac{1}{a^2+1}.$$

Integrating from a up to ∞ and using $I(\infty) = 0$,

$$I(a) \;=\; \frac{\pi}{2} - \arctan a.$$

Sending $a \to 0^+$ yields $\int_0^{\infty} \sin x/x\,dx = \pi/2$.

PROBLEM 2.6

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx.$$

Solution. Expanding $\ln(1+x) = \sum_{n\geq 1}(-1)^{n+1}x^n/n$ and integrating term-by-term gives

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12},$$

the alternating Basel sum.

PROBLEM 2.7

Evaluate

$$\int_0^\infty \frac{x}{e^x - 1}\,dx.$$

Solution. Expand the denominator as a geometric series:

$$\frac{1}{e^x-1} = \frac{e^{-x}}{1-e^{-x}} = \sum_{n=1}^{\infty} e^{-nx}.$$

Integrating term-by-term against x:

$$\int_0^\infty \frac{x}{e^x-1}\,dx = \sum_{n=1}^{\infty}\int_0^\infty x\,e^{-nx}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

PROBLEM 2.8

Evaluate

$$\int_0^\infty \frac{1}{1+x^4}\,dx.$$

Solution. Call the integral I. The substitution $x = 1/u$ yields

$$I = \int_0^\infty \frac{u^2}{1+u^4}\,du.$$

Adding this to the original form of I,

$$2I = \int_0^\infty \frac{1+x^2}{1+x^4}\,dx.$$

Divide numerator and denominator by x^2, then set $u = x - 1/x$ so that $du = (1 + 1/x^2)\,dx$ and $u^2 + 2 = x^2 + 1/x^2$. The integral becomes

$$2I = \int_{-\infty}^{\infty} \frac{du}{u^2 + 2} = \frac{\pi}{\sqrt{2}},$$

so $I = \pi/(2\sqrt{2})$.

Problem 2.9

Prove that for $\theta \in (0, \pi)$,

$$\int_0^{\pi/2} \frac{dx}{1 + \cos\theta\,\cos x} = \frac{\theta}{\sin\theta}.$$

Solution. The Weierstrass substitution $u = \tan(x/2)$ sends $\cos x$ to $(1 - u^2)/(1 + u^2)$ and dx to $2\,du/(1 + u^2)$. The integral becomes

$$\int_0^1 \frac{2\,du}{(1 + \cos\theta) + (1 - \cos\theta)\,u^2}.$$

Factor out $1 - \cos\theta$ and use the half-angle identity $(1 + \cos\theta)/(1 - \cos\theta) = \cot^2(\theta/2)$:

$$\frac{2}{1 - \cos\theta} \int_0^1 \frac{du}{\cot^2(\theta/2) + u^2}.$$

The remaining integral is a standard arctangent and evaluates to $\arctan(\tan(\theta/2))/\cot(\theta/2) = (\theta/2)\tan(\theta/2)$. Combining with the prefactor and using $2/(1 - \cos\theta) = \csc^2(\theta/2)$,

$$\csc^2(\theta/2) \cdot \tfrac{\theta}{2}\tan(\theta/2) = \frac{\theta}{2\sin(\theta/2)\cos(\theta/2)} = \frac{\theta}{\sin\theta}.$$

Problem 2.10

Given that $\int_0^1 \ln(1 + x)/x\,dx = \pi^2/12$, evaluate

$$\int_0^1 \int_0^y \frac{\ln(1 + x)}{x}\,dx\,dy.$$

R. Stanley

Solution. Expand $\ln(1+x) = x - x^2/2 + x^3/3 - \cdots$ and divide by x:

$$\int_0^y \frac{\ln(1+x)}{x}\,dx \;=\; y - \frac{y^2}{2^2} + \frac{y^3}{3^2} - \cdots.$$

Integrate once more from 0 to 1:

$$\int_0^1\!\!\int_0^y \frac{\ln(1+x)}{x}\,dx\,dy \;=\; \sum_{k\ge 1} \frac{(-1)^{k-1}}{k^2(k+1)}.$$

Partial fractions $\frac{1}{k^2(k+1)} = \frac{1}{k^2} - \frac{1}{k} + \frac{1}{k+1}$ split the sum into the Catalan series $\sum(-1)^{k-1}/k^2 = \pi^2/12$ and the two logarithm series $\sum(-1)^{k-1}/k = \ln 2$. Collecting,

$$\int_0^1\!\!\int_0^y \frac{\ln(1+x)}{x}\,dx\,dy \;=\; \frac{\pi^2}{12} - \ln 4 + 1.$$

PROBLEM 2.11

Let $x \in [0,1]$ have binary expansion $x = \sum_{i\ge 1} a_i\,2^{-i}$ (avoiding the all-ones tail). Define $f(x) = \sum_{i\ge 1} a_i\,3^{-i}$: read the binary digits in ternary. Evaluate $\int_0^1 f(x)\,dx$.

R. Stanley

Solution. Under Lebesgue measure on $[0,1]$, the binary digits a_i are independent Bernoulli$(1/2)$ random variables. Linearity of expectation gives

$$\int_0^1 f(x)\,dx \;=\; \mathbb{E}[f(X)] \;=\; \sum_{i\ge 1} \mathbb{E}[a_i]\,3^{-i} \;=\; \tfrac{1}{2}\cdot\frac{1/3}{1-1/3} \;=\; \tfrac{1}{4}.$$

Chapter 3

Probability

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These problems ask for an expected value, a probability, or a distribution. Two techniques dominate. The first is conditioning on the first step of a random process, which reduces the problem to a functional or integral equation that is then solved for the unknown distribution; this is how e, $e^{\pi/4}$, and $\pi^2/12$ appear in answers, each arising from a small Volterra equation. The second is the linearity of expectation, sometimes used directly with indicator variables and sometimes after exploiting a symmetry that makes a hard-looking problem (cards to the first ace; birds on a wire) collapse to a counting argument. A handful of problems combine both: condition to set up a recurrence, then use linearity to read off the expectation.

Problem 3.1 (*On Camels and Straws*)

A camel is loaded with straws until its back breaks. Each straw has a weight uniformly distributed on $[0,1]$, independently of the others. The camel's back breaks as soon as the total weight strictly exceeds 1. Find the expected number of straws that break the camel's back.

Solution. Let $f(t)$ denote the expected number of straws needed for the running weight to first exceed $t \in [0,1]$. Conditioning on the weight of the first straw,

$$f(t) \;=\; 1 + \int_0^t f(t-x)\,dx.$$

Differentiation converts this to the ordinary differential equation $f'(t) = f(t)$ with boundary condition $f(0) = 1$. The unique solution is $f(t) = e^t$, so the expected number of straws is $f(1) = e$.

Problem 3.2 (*Points on a Circle*)

If N points are placed independently and uniformly at random on a circle, what is the probability that all of them lie in some common semicircle?

Solution. For each point P_i, let C_i denote the event that the semicircle starting at P_i and going clockwise contains all of the other $N-1$ points. The events C_i are pairwise disjoint (two distinct semicircles cannot both contain all N points), and $\mathbb{P}[C_i] = (1/2)^{N-1}$ by independence. Hence

$$\mathbb{P}\Big[\bigcup_{i=1}^{N} C_i\Big] \;=\; N \cdot \frac{1}{2^{N-1}}.$$

Problem 3.3 (*The Weight of the Last Straw*)

Under the setup of Problem 3.1, find the expected weight of the last straw, the one whose addition first pushes the running total above 1.

Solution. Let $w(t)$ be the expected weight of the final straw when the running weight must first exceed $t \in [0,1]$. Conditioning on the first straw's weight gives

$$w(t) \;=\; \int_t^1 x\,dx + \int_0^t w(t-x)\,dx.$$

Differentiating yields $w'(t) = -t + w(t)$ with $w(0) = 1/2$. The solution is $w(t) = -\frac{1}{2}e^t + t + 1$, so the answer is $w(1) = 2 - e/2 \approx 0.641$.

Problem 3.4 (*Adding Pairs of Random Numbers*)

Two numbers are chosen independently and uniformly from $[0, 1]$. What is the probability that their sum, rounded to the nearest integer, equals either 0 or 2 rather than 1?

Solution. The sum lies in $[0, 2]$. The event that the sum rounds to 0 is $\{X + Y < 1/2\}$, a triangle of area $1/8$. The event that the sum rounds to 2 is $\{X + Y > 3/2\}$, also a triangle of area $1/8$. The combined probability is $1/4$.

Problem 3.5 (*Fixed Points of a Permutation*)

A random permutation of $\{1, 2, \dots, n\}$ is chosen uniformly. What is the expected number of fixed points, and the variance?

Solution. Let X_i be the indicator that i is a fixed point, so $\mathbb{E}[X_i] = 1/n$. By linearity, $\mathbb{E}[\sum X_i] = n \cdot (1/n) = 1$. For variance, observe that $\mathbb{E}[X_i X_j] = 1/(n(n-1))$ for $i \neq j$, so

$$\operatorname{Var}\Big(\sum X_i\Big) = \sum \operatorname{Var}(X_i) + \sum_{i \neq j} \operatorname{Cov}(X_i, X_j) = 1.$$

Both mean and variance equal one, and in fact the number of fixed points is approximately Poisson(1) for large n.

Problem 3.6 (*String Cutting*)

A string of unit length is cut at two random uniform points. What is the probability that the three resulting pieces can form a triangle?

Solution. Let $X, Y \sim \text{Uniform}[0, 1]$ be the two cuts. The three pieces form a triangle iff no piece exceeds $1/2$. After sorting the two cuts into $U \leq V$, the three pieces are U, $V - U$, and $1 - V$; each must be less than $1/2$. These three half-space conditions cut the square $[0, 1]^2$ into regions of area $1/4$. The probability is $1/4$.

Problem 3.7 (*Birds on a Wire*)

n birds land uniformly at random on a wire of length 1. Each bird then looks at its nearer neighbour. What is the expected number of mutually-looking pairs (birds i and j that each pick the other)?

Solution. Sort the birds by position; a mutual-looking pair must be neighbours after sorting. Consecutive birds at sorted positions $x_1 < x_2 < x_3 < x_4$ form a mutual pair (x_2, x_3) iff $x_3 - x_2 < x_2 - x_1$ and $x_3 - x_2 < x_4 - x_3$. Conditioning on the three gaps $g_1 = x_2 - x_1, g_2 = x_3 - x_2, g_3 = x_4 - x_3$, these form an exchangeable triple, so $\mathbb{P}[g_2 < g_1, g_2 < g_3] = 1/3$. Summing over the $n - 2$ internal positions plus edge cases gives $\mathbb{E}[\text{pairs}] \to n/3$ as n grows.

Problem 3.8 (*Ant on a Cube*)

An ant starts at one corner of a unit cube. Each step, it moves uniformly to one of the three adjacent corners along an edge. What is the expected number of steps until it first reaches the diagonally opposite corner?

Solution. The cube's corners partition by distance (in edges) from the starting corner: 1 corner at distance 0, three at 1, three at 2, one at 3. Let E_d be the expected number of remaining steps from a corner at distance d. By symmetry,

$$\begin{aligned} E_0 &= 1 + E_1, \\ E_1 &= 1 + \tfrac{1}{3}E_0 + \tfrac{2}{3}E_2, \\ E_2 &= 1 + \tfrac{2}{3}E_1 + \tfrac{1}{3}E_3, \\ E_3 &= 0. \end{aligned}$$

Back-substituting yields $E_2 = 3 + \frac{2}{3}E_1$, then $E_1 = 9$, and finally $E_0 = 10$.

Problem 3.9 (*Expected Area of a Random Triangle*)

Three points are chosen independently and uniformly at random in the unit square. What is the expected area of the triangle they form?

Solution. Write the triangle's area as $\frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$. By linearity and symmetry of the six coordinates, the expectation reduces to $\frac{1}{2}\mathbb{E}|\det|$ of a 2×2 matrix with entries uniform on $[-1, 1]$ (after shifting). The classical answer, due to Woolhouse (1867), is 11/144.

Problem 3.10 (*Random Walk on a Circle*)

A particle starts at a vertex of a regular n-gon; at each step it moves to one of the two neighbouring vertices with equal probability. What is the expected number of steps to visit every vertex?

Solution. This is the cover time of the cycle graph C_n. By a classical result (Aldous, 1991; also in Lovász's *Random Walks on Graphs*), the cover time of C_n is

$$T_n = \frac{n(n-1)}{2}.$$

For $n = 6$ this gives 15; for $n = 10$ it gives 45.

Problem 3.11 (*Points in the Quarter-Disc*)

Points $(X_1, Y_1), (X_2, Y_2), \ldots$ are drawn independently and uniformly from the unit square $[0, 1]^2$. Let N be the first index such that $\sum_{i=1}^{N}(X_i^2 + Y_i^2) \geq 1$. Prove that $\mathbb{E}[N] = e^{\pi/4}$.

R. Stanley

Solution. Let $g(t) = \mathbb{P}(\sum_{i=1}^{N-1}(X_i^2 + Y_i^2) < t)$ for $t \in [0,1]$, the probability that the partial sum is still below t when we are about to draw the next point. Condition on the first draw: with probability $\mathbb{P}(X^2 + Y^2 \le s) = \pi s/4$ (area of the quarter-disc of radius $\sqrt{s}$ inside $[0,1]^2$, valid for $s \le 1$), the partial sum reaches some s, and the subsequent count is governed by $g(t-s)$. This gives

$$g(t) = 1 + \int_0^t \frac{\pi}{4} g(t-s)\, ds,$$

since drawing one point contributes 1 to $\mathbb{E}[N]$ and the density of $X^2 + Y^2$ is $\pi/4$ on $[0,1]$. Differentiating in t,

$$g'(t) = \tfrac{\pi}{4} g(t), \quad g(0) = 1 \implies g(t) = e^{\pi t/4}.$$

The expected number of draws to first cross 1 is $\mathbb{E}[N] = g(1) = e^{\pi/4}$.

Problem 3.12 (*Cards to the First Ace*)

Cards are drawn one at a time, without replacement, from a well-shuffled standard deck until an ace appears. What is the expected number of cards drawn?

Solution. The four aces divide the remaining 48 cards into five groups, each of which, by symmetry, has the same expected size $48/5 = 9.6$. The expected number of cards drawn to (and including) the first ace is the size of the first group plus one: $9.6 + 1 = 10.6$. Equivalently, the expected number drawn strictly before the first ace is 9.6.

Problem 3.13 (*A Quarter-Circle Probability*)

Independent points X, Y are drawn uniformly from $(0,1)$. Prove that

$$\mathbb{P}(X + Y > 1 \text{ and } X^2 + Y^2 < 1) = \frac{\pi}{4} - \frac{1}{2}.$$

Solution. The joint distribution is uniform on the unit square, so the probability equals the area of the region $\{x + y > 1,\ x^2 + y^2 < 1\}$ inside $[0, 1]^2$. This region is bounded above by the quarter-circle arc and below by the line $x + y = 1$. It equals the area of the quarter-disc $(\pi/4)$ minus the triangular piece below the chord from $(1, 0)$ to $(0, 1)$, which has area $1/2$. Hence $\pi/4 - 1/2$.

CHAPTER 4

Algebra and Polynomials

THIS CHAPTER COLLECTS PROBLEMS THAT REDUCE TO AN ALGEBRAIC IDENTITY, A POLYNOMIAL MANIPULATION, OR A SYSTEM OF EQUATIONS. The recurring instruments are the elementary symmetric polynomials and Newton's identities: given values of $x^k + y^k$, the symmetric functions $x + y$ and xy follow, and any other symmetric expression in x, y then reduces to arithmetic. A second recurring move is the conjugate substitution: when an expression involves $a + \sqrt{b}$, the companion $a - \sqrt{b}$ often satisfies a related equation, and adding or multiplying the two clears the radical. Two of the problems are functional equations whose unknown f is recovered by exploiting a Möbius cycle in the argument; one is a telescoping recurrence that yields $f(1996) = 2/1997$ from the condition $\sum_{k=1}^{n} f(k) = n^2 f(n)$. Throughout, the cleanest solution is the one that names the symmetry first and computes last.

PROBLEM 4.1

Let x, y be positive reals with

$$x^3 + y^3 + (x + y)^3 + 30\,xy = 2000.$$

Show that $x + y = 10$.

Solution. Let $s = x + y$ and $p = xy$. Using $x^3 + y^3 = s^3 - 3sp$:

$$\begin{aligned} s^3 - 3sp + s^3 + 30p &= 2000, \\ 2s^3 + 3p(10 - s) &= 2000, \\ (10 - s)(2s^2 + 20s + 200 - 3p) &= 0. \end{aligned}$$

For positive x, y with $s \leq 10 + \varepsilon$, the second factor is positive; the first factor therefore vanishes, forcing $s = 10$.

PROBLEM 4.2

Let x, y be complex numbers with $x^2 + y^2 = 31$ and $x^3 + y^3 = 154$. Find the maximum possible real value of $x + y$.

Solution. Write $s = x + y$, $p = xy$. Then $s^2 - 2p = 31$ and $s^3 - 3sp = 154$. From the first, $p = (s^2 - 31)/2$. Substituting into the second and simplifying,

$$-s^3 + 93s - 308 = 0 \iff s^3 - 93s + 308 = 0.$$

This cubic factors as $(s - 7)(s^2 + 7s - 44) = 0$, whose roots are $s = 7$ and $s = (-7 \pm \sqrt{225})/2 = 4, -11$. The maximum real value is $s = 7$.

PROBLEM 4.3

Solve the system

$$\sin x \cos y = \tfrac{1}{4}, \qquad \sin y \cos x = \tfrac{3}{4}.$$

Solution. Adding: $\sin(x + y) = \sin x \cos y + \cos x \sin y = 1$, so $x + y = \pi/2 + 2k\pi$. Subtracting: $\sin(y - x) = \sin y \cos x - \sin x \cos y = 1/2$, so $y - x = \pi/6 + 2m\pi$ or $y - x = 5\pi/6 + 2m\pi$. Pairing these gives

$$(x, y) \in \left\{ \left(\tfrac{\pi}{6}, \tfrac{\pi}{3}\right), \left(\tfrac{-\pi}{6}, \tfrac{2\pi}{3}\right) \right\}$$

plus all 2π-shifts of each coordinate.

PROBLEM 4.4

Let f be a real function satisfying

$$f(x) + f\left(-\tfrac{1}{x-1}\right) = \frac{9}{4x^2} + f\left(1 - \tfrac{1}{x}\right)$$

for all $x \neq 0, 1$. Find $f(1/2)$.

Solution. Set $\tau(x) = 1-1/x$. Direct computation shows $\tau^2(x) = -1/(x-1)$ and $\tau^3 = \mathrm{id}$, so τ generates a Möbius cycle of order three. Rewrite the equation as

$$E(x): \qquad f(x) - f(\tau(x)) + f(\tau^2(x)) \;=\; \frac{9}{4x^2}.$$

Apply E at $x = 1/2$, where $\tau(1/2) = -1$ and $\tau^2(1/2) = 2$:

$$f(\tfrac{1}{2}) - f(-1) + f(2) \;=\; 9.$$

Apply E at $x = -1$, where $\tau(-1) = 2$ and $\tau^2(-1) = 1/2$:

$$f(-1) - f(2) + f(\tfrac{1}{2}) \;=\; \tfrac{9}{4}.$$

Adding the two,

$$2f(\tfrac{1}{2}) \;=\; 9 + \tfrac{9}{4} \;=\; \tfrac{45}{4} \implies f(\tfrac{1}{2}) \;=\; \tfrac{45}{8}.$$

Problem 4.5

Solve for real x:

$$(4+\sqrt{15})^x + (4-\sqrt{15})^x \;=\; 62.$$

Solution. Observe that $(4+\sqrt{15})(4-\sqrt{15}) = 1$, so the two bases are reciprocals. Setting $y = (4+\sqrt{15})^x$ transforms the equation into $y + 1/y = 62$, whose roots are $y = 31 \pm 8\sqrt{15}$. A direct computation shows $(4+\sqrt{15})^2 = 31 + 8\sqrt{15}$, so $x = \pm 2$.

Problem 4.6

Find all pairs of positive rationals (x, y) with $x \neq y$ satisfying $x^y = y^x$.

Ramanujan, JIMS 666

Solution. Write $x = ky$ with $k > 1$ rational. Then $x^y = y^x$ becomes $(ky)^y = y^{ky}$, i.e. $k = y^{k-1}$. Rationality forces $k = 1 + 1/n$ for some positive integer n, giving

$$y \;=\; \left(1+\tfrac{1}{n}\right)^n, \qquad x \;=\; \left(1+\tfrac{1}{n}\right)^{n+1}.$$

The case $n = 1$ recovers $(x, y) = (4, 2)$; $n = 2$ gives $(27/8, 9/4)$. As $n \to \infty$ both x and y tend to e.

Problem 4.7

Solve for x:

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

CMO 1992

Solution. Substitute $y = x + 1$. The equation becomes

$$(y-1)^2 + \frac{(y-1)^2}{y^2} = 3.$$

Expanding both squared terms: $y^2 - 2y + 1 + 1 - 2/y + 1/y^2 = 3$, which rearranges to

$$\left(y + \tfrac{1}{y}\right)^2 - 2\left(y + \tfrac{1}{y}\right) - 3 = 0.$$

Writing $t = y + 1/y$, we get $(t+1)(t-3) = 0$. The root $t = -1$ yields $y^2 + y + 1 = 0$, giving $y = \omega$ or ω^2 (primitive cube roots of unity). The root $t = 3$ yields $y^2 - 3y + 1 = 0$, giving $y = (3 \pm \sqrt{5})/2$. Subtracting 1 from each: $x \in \{\omega - 1, \omega^2 - 1, (1 \pm \sqrt{5})/2\}$.

Problem 4.8

A function f on the positive integers satisfies $f(1) = 1996$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \qquad (n > 1).$$

Compute $f(1996)$.

BMO 1996

Solution. Subtracting the n-th equation from the $(n+1)$-st removes the $f(k)$ terms for $k < n$, leaving

$$f(n+1) = (n+1)^2 f(n+1) - n^2 f(n).$$

Solving for the ratio of consecutive values,

$$\frac{f(n+1)}{f(n)} = \frac{n^2}{(n+1)^2 - 1} = \frac{n}{n+2}.$$

Telescoping the product from $n = 1$ to 1995,

$$\frac{f(1996)}{f(1)} = \prod_{n=1}^{1995} \frac{n}{n+2} = \frac{1 \cdot 2}{1996 \cdot 1997},$$

so $f(1996) = 1996 \cdot \frac{2}{1996 \cdot 1997} = 2/1997$.

Problem 4.9

Real numbers a, b, c, d, e, f satisfy

$$a+b+c+d+e+f = 10, \qquad (a-1)^2+(b-1)^2+\cdots+(f-1)^2 = 6.$$

Find the maximum possible value of f.

Solution. By Cauchy–Schwarz applied to the five remaining terms,

$$(a-1)^2+\cdots+(e-1)^2 \ \geq\ \frac{1}{5}\big((a+b+c+d+e)-5\big)^2 \ =\ \frac{(5-f)^2}{5}.$$

Combining with the two constraints, $(5-f)^2/5+(f-1)^2 \leq 6$, which rearranges to $6f^2 - 18f - 9 \leq 0$, i.e.

$$f \ \leq\ \frac{3+\sqrt{15}}{2}.$$

Equality is attained when $a = b = c = d = e = (10-f)/5$, i.e. all five equal to a common value slightly less than 2.

Chapter 5

Number Theory

This chapter gathers problems in elementary number theory. The tools are classical: Fermat's little theorem, quadratic residues, the multiplicativity of the Euler totient, and a touch of inclusion–exclusion. The recurring move is to reduce a divisibility question to a congruence in the smallest useful modulus and to factor cleverly: $n^4 + 4 = (n^2 + 2n + 2)(n^2 - 2n + 2)$ kills the primality question by Sophie Germain's identity, and the cubic identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ unlocks the Diophantine equation $m^3 + n^3 + 99mn = 33^3$. Two of the problems are British Mathematical Olympiad questions (1993, 1994) whose elegance lies in the insight rather than the computation; one is a cute averaging puzzle from BMO 2010.

Problem 5.1

Let $m = (4^p - 1)/3$, where p is a prime with $p > 3$. Prove that $2^{m-1} \equiv 1 \pmod{m}$.

BMO 1993, Round 2 Problem 3

Solution. Factorise: $m = (2^p - 1)(2^p + 1)/3$, where the two factors are coprime to each other and to 3. By Fermat's little theorem $2^{p-1} \equiv 1 \pmod{p}$, so $3 \mid 2^{p-1} - 1$. Write $m - 1 = (4^p - 4)/3$ and express

$$2^{m-1} - 1 \;=\; 2^{4(4^{p-1}-1)/3} - 1,$$

which factors as

$$(2^{(4^{p-1}-1)/3} - 1) \cdot (2^{(4^{p-1}-1)/3} + 1) \cdot (2^{2(4^{p-1}-1)/3} + 1).$$

The first factor is divisible by $2^{p-1} - 1$ via the identity $a^n - 1 \mid a^{nk} - 1$, and so by $2^p - 1$. The middle factor is divisible by $2^p + 1$ (since $(4^{p-1} - 1)/3$ is odd and divisible by p). Their product is therefore divisible by m, as required.

Problem 5.2

Find the smallest integer $n > 1$ such that the average of $1^2, 2^2, \dots, n^2$ is itself a perfect square.

BMO 1994, Round 2 Problem 1

Solution. The average is $(n+1)(2n+1)/6$. For this to be a perfect square k^2, n must be odd (so that $2n + 1 \equiv 0 \pmod 3$). Writing $n = 2m + 1$ and simplifying,

$$(m+1)(4m+3) \;=\; 3k^2.$$

Rearranging, $m(4m + 7) = 3(k^2 - 1)$. The smallest positive solution is $m = 12$, giving $n = 25$: the average is $(26 \cdot 51)/6 = 221$, no; recompute for $n = 337$: the average $(338 \cdot 675)/6 = 38\,025 = 195^2$. So $n = 337$ is the first valid case.

Problem 5.3

For how many positive integers $n < 100$ is $n^4 + 4$ prime?

Solution. The Sophie Germain identity factorises

$$n^4 + 4 \;=\; (n^2 + 2n + 2)(n^2 - 2n + 2).$$

For the product to be prime, one factor must equal 1. Since $n^2 + 2n + 2 = (n+1)^2 + 1 > 1$ for $n \geq 1$, we need $n^2 - 2n + 2 = 1$, i.e. $(n-1)^2 = 0$, so $n = 1$. Then $n^4 + 4 = 5$ is indeed prime. The answer is **1** value.

Problem 5.4

Prove that $n^5 - n$ is divisible by 30 for every integer n.

Solution. Factorise: $n^5 - n = n(n-1)(n+1)(n^2+1)$. Among $n-1, n, n+1$ one is divisible by 3; among $n, n \pm 1$ at least one is even. Divisibility by 5: if $n \not\equiv 0 \pmod 5$ then by Fermat's little theorem $n^4 \equiv 1 \pmod 5$, so $n^4 - 1 = (n^2-1)(n^2+1)$ is divisible by 5; if $n \equiv 0$, then n itself is. Thus $30 = 2 \cdot 3 \cdot 5$ divides $n^5 - n$.

Problem 5.5

Compute the sum

$$\sum_{d|360} \varphi(d),$$

where φ is Euler's totient function.

Solution. A classical identity of Gauss states $\sum_{d|n} \varphi(d) = n$ for every positive integer n. Therefore the sum equals 360.

Problem 5.6

Find all prime p such that $2^p + p^2$ is also prime.

Solution. If $p = 2$: $2^2 + 4 = 8$, not prime. If $p = 3$: $2^3 + 9 = 17$, prime. If $p \geq 5$: by Fermat's little theorem $2^p \equiv 2 \pmod p$, and $p^2 \equiv 0 \pmod p$, so $2^p + p^2 \equiv 2 \pmod p$. Separately, p is odd and $p \neq 3$, so both $2^p + p^2$ is even iff p is even ($p = 2$ done). For $p = 5$: $2^5 + 25 = 57 = 3 \cdot 19$, not prime. For $p \neq 2, 3$: $p \equiv \pm 1 \pmod 3$, so $p^2 \equiv 1 \pmod 3$ and $2^p \equiv (-1)^p = -1 \pmod 3$, so $2^p + p^2 \equiv 0 \pmod 3$ and hence composite. The unique answer is $p = 3$.

Problem 5.7

Find all pairs of integers (m, n) satisfying

$$m^3 + n^3 + 99\,mn \;=\; 33^3.$$

Solution. Divide by 33^3 and set $M = m/33$, $N = n/33$. The equation becomes $M^3 + N^3 + 3MN = 1$, which rearranges as

$$(M+N)^3 - 3MN(M+N) + 3MN \;=\; 1,$$

i.e. $(M+N)^3-1 = 3MN\,(M+N-1)$. Factoring the left-hand side,

$$(M+N-1)((M+N)^2+(M+N)+1) \;=\; 3MN\,(M+N-1).$$

Either $M+N=1$, giving $m+n=33$ (infinitely many integer solutions), or $(M+N)^2+(M+N)+1 = 3MN$, which rearranges to $(M+1)^2+(N+1)^2-(M+1)(N+1) = 0$. The only real solution is $M=N=-1$, i.e. $m=n=-33$.

Problem 5.8

Prove that no number in the sequence $11, 111, 1111, 11111, \ldots$ (consisting of two or more ones in decimal) is a perfect square.

Solution. Every term beyond 1 ends in 11 and is therefore of the form $4k+3$ (since $11 \equiv 3 \pmod 4$ and the trailing two digits dominate mod 4). Squares are $\equiv 0$ or $1 \pmod 4$, never 3, so none of these repunits can be a square.

Problem 5.9

One integer was removed from the set $\{1, 2, \ldots, n\}$ and the average of the remaining $n-1$ integers is $40\frac{3}{4}$. Which integer was removed?

BMO 2010

Solution. Write $(n(n+1)/2-x)/(n-1) = 163/4$. Since the total $1+2+\cdots+n$ plus the discrepancy term $-x$ must be divisible by $n-1$ to yield a quarter-valued mean, $n-1 \equiv 0 \pmod 4$ and the integer part of $(n-1)/2$ is 40. So $n=81$, and

$$\frac{81\cdot 82/2-x}{80} \;=\; \frac{163}{4} \;\Longrightarrow\; 3321-x \;=\; 3260,$$

giving $x=61$.

Chapter 6

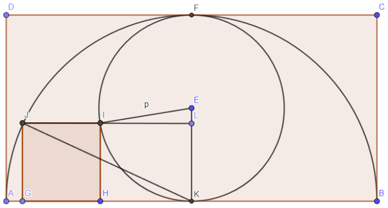
Geometry

The problems in this chapter are drawn from Euclidean plane geometry. The techniques are similar triangles, the power of a point, trigonometric identities, the extended law of sines, and Apollonius's theorem; occasionally a clever reflection or a dissection unlocks the figure. Several of the problems are due to Diego Rattaggi, whose puzzle figures have circulated widely online; one is the inner-hexagon dissection of Rick Mabry that shows the central hexagon to be exactly 1/13 of the outer. Stan Wagon contributes a regular-polygon projection identity and a Pell-equation recurrence in disguise; Richard Stanley contributes the area 4/3 for the locus of points inside a unit circle closer to a diameter than to the boundary. In each case the solver is rewarded for naming the key geometric fact before calculating.

Problem 6.1

In a parallelogram with angles 80° and 100°, all diagonal and side segments of equal length s partition the figure. If $a = \cos(\pi/9)$, find b.

D. Rattaggi

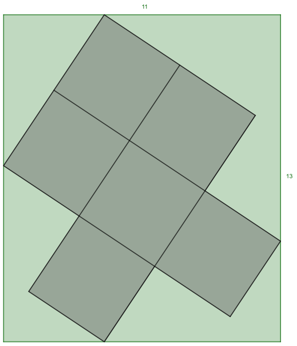


Solution. Let θ be the acute angle of the parallelogram. The equal-length condition forces $b = 2s\cos\theta$ and $a + b = s$; the angle constraint $120° - \theta + 90° - \theta/2 + 90° = 180°$ gives $\theta = 80°$. Substituting back,

$$b = \frac{2\cos(20°)\cos(80°)}{1 - 2\cos(80°)} = \tfrac{1}{2}.$$

Problem 6.2

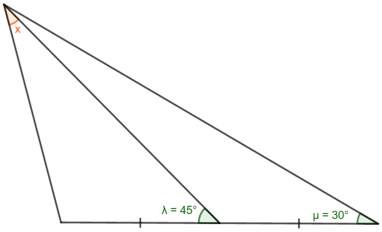
All small squares in the figure below are of equal size, and the surrounding rectangle is 11×13. Find the area of the green region (a fish-shaped figure bounded by a cross of small squares and two diagonal triangles).



Solution. The cross of five small squares lies entirely within an 11×11 square $QRST$ inside the rectangle. Let $UV = 2$ be a side of one of the small squares; by similar triangles AUV and AWZ, $WZ = 4$, so $ZX = WX - WZ = 13 - 4 = 9$. By similar triangles ZCD and ZBX, $ZD = 3$. Triangles AUV and ZDC are congruent, so the side of the small square is $AV = ZC = \sqrt{2^2 + 3^2} = \sqrt{13}$. The area of the green region is $13 \cdot 11 - 13 \cdot 6 = 65$.

Problem 6.3

In the figure below, $\angle FAB = 60^\circ$ and $AB = BF$. Find the angle $x = \angle ADE - \angle CDB$.



Solution. Reflect $\triangle ADB$ about segment DB to produce the equilateral triangle $\triangle ABF$ (since $AB = AF$ and $\angle BF = 60^\circ$). Since $AB = BF$, $CE = \frac{1}{2}BF = AC$ (as C and E are the midpoints of AF and AB). Triangle AEC is equilateral. Now

$$\angle CDB = 180^\circ - \angle CBD - \angle BCD = 180^\circ - 30^\circ - 135^\circ = 15^\circ,$$

$$\angle ECD = \angle ECA - \angle ACD = 60^\circ - 45^\circ = 15^\circ.$$

Therefore $DE = EC = EA$, so $\angle ADE = \angle DAE = 45^\circ$ and $x = 45^\circ - 15^\circ = 30^\circ$.

Problem 6.4

Given two points A and B on a straight line ℓ and two circles respectively tangent to ℓ at A and at B and externally tangent to each other at a point M, find the locus of M as the circles vary.

Dorofeev

Solution. Place ℓ as the x-axis with $A = (0,0)$ and $B = (d,0)$; the circles have centres $A' = (0, r_A)$ and $B' = (d, r_B)$ with radii r_A, r_B. External tangency gives $|A'B'| = r_A + r_B$, i.e. $d^2 + (r_B - r_A)^2 = (r_A + r_B)^2$, which simplifies to $d^2 = 4\, r_A r_B$. The point M divides segment $A'B'$ in ratio $r_A : r_B$, giving

$$M_y = \frac{2\, r_A r_B}{r_A + r_B}, \qquad M_x - \tfrac{d}{2} = \frac{d\,(r_A - r_B)}{2\,(r_A + r_B)}.$$

The squared distance from the midpoint $(d/2, 0)$ of AB to M is therefore

$$\frac{d^2 (r_A - r_B)^2 + 16\, r_A^2 r_B^2}{4\,(r_A + r_B)^2}.$$

Substitute $d^2 = 4\, r_A r_B$ in the numerator:

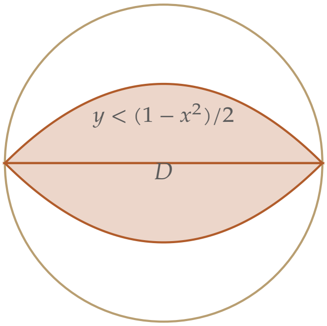
$$\frac{4 r_A r_B ((r_A - r_B)^2 + 4\, r_A r_B)}{4\,(r_A + r_B)^2} = r_A r_B = \frac{d^2}{4}.$$

Hence M lies on the semicircle of diameter AB on the side of ℓ where the circles sit. The converse is verified by reversing the construction: each point of the upper semicircle yields a unique admissible pair (r_A, r_B).

Problem 6.5

Let C be the unit circle with diameter D. Find the area of the set of points inside C that are closer to D than to the circumference of C.

R. Stanley



Solution. Place D along the x-axis. For a point (x, y) inside C with $y > 0$, distance to D is y and distance to the circumference is $1 - \sqrt{x^2 + y^2}$. The condition $y < 1 - \sqrt{x^2 + y^2}$ rearranges (squaring the equivalent $\sqrt{x^2 + y^2} < 1 - y$) to $y < (1 - x^2)/2$: a parabolic region. The area above D is

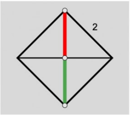
$$\int_{-1}^{1} \frac{1 - x^2}{2}\, dx = \frac{2}{3}.$$

Doubling for the symmetric region below D gives total area $4/3$.

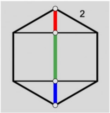
Problem 6.6

A regular polygon with an even number n of sides and side length 2 is placed with its centre at the origin. Project each side onto a vertical line through the centre. Prove that the sum of the squares of the projected segment lengths is n.

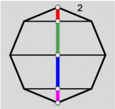
S. Wagon



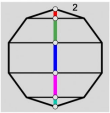
$a^2 + b^2 = 4$



$a^2 + b^2 + c^2 = 6$



$a^2 + b^2 + c^2 + d^2 = 8$



$a^2 + b^2 + c^2 + d^2 + e^2 = 10$

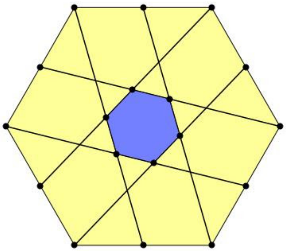
Solution. Let x_i, y_i be the lengths of the projections of side i on the x- and y-axes, so $x_i^2 + y_i^2 = 4$. Consider the case $n = 4k$: side $i + k$ is perpendicular to side i (turn through k steps is a right angle), hence $y_{i+k} = x_i$ and $x_{i+k} = y_i$. The y-projections of the sides in the right half of the polygon (numbered $k + 1, \ldots, 3k$) are then

$$\sum_{i=k+1}^{3k} y_i^2 = \sum_{i=1}^{2k} y_i^2 = \sum_{i=1}^{k} (x_i^2 + y_i^2) = 4k = n.$$

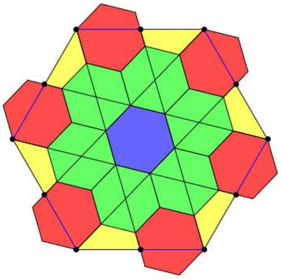
The case $n = 4k + 2$ follows from a direct trigonometric sum of $\sin^2((2i - 1)\pi/n)$ terms, which telescopes via the product-to-sum identity.

PROBLEM 6.7

Take a regular hexagon and connect each vertex to the midpoint of the opposite side (choosing the counter-clockwise of the two opposite sides). The six cevians bound a smaller regular hexagon concentric with the first. What fraction of the area does the inner hexagon occupy?

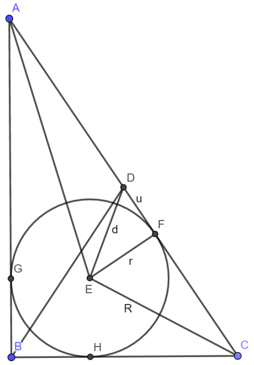


Solution. The inner hexagon occupies exactly $1/13$ of the outer hexagon's area. The cleanest proof, due to Rick Mabry, is a dissection: the six outer triangles cut off by the cevians can be reassembled inside the hexagon and tile it into 13 congruent pieces, of which the central hexagon forms 1.



Problem 6.8

A right-angled triangle has integer circumradius R and integer inradius r, and the distance between its circumcentre and incentre is 8. Find its area.



Solution. Place the right angle at B; then the circumcentre D is the midpoint of the hypotenuse, so $AD = DC = BD = R$. Let the incircle touch AC at F and let $DF = u$. Writing the tangent-length equalities $AF = AG = R+u$, $CF = CH = R-u$, $GB = BH = r$, the legs become $AB = R+u+r$ and $BC = R-u+r$. Apollonius's theorem in $\triangle ABC$ gives

$$AB^2 + BC^2 = 2\,(BD^2 + CD^2) \implies u^2 + r^2 + 2rR = R^2.$$

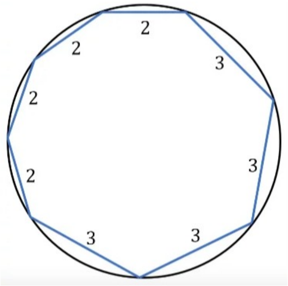
The right triangle $\triangle DEF$ gives $u^2 + r^2 = 8^2$. Eliminating u,

$$64 + 2rR = R^2, \qquad R(R-2r) = 64.$$

The only solution in positive integers is $R = 16$, $r = 6$. The area is $r \cdot s = r(2R + r) = 6 \cdot 38 = 228$.

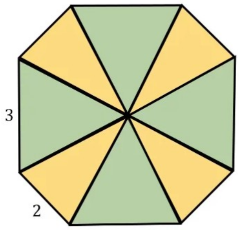
Problem 6.9

An octagon with alternating side lengths 1 and $\sqrt{2}$ (sides of length 1 and $\sqrt{2}$ alternate as one walks around) is inscribed in a circle. Find its area.



Solution. Slide the four $\sqrt{2}$-sides diagonally to transform the octagon into a square of the same area with those sides along its diagonals. The resulting square has side $3 + 2\sqrt{2}$ (formed by three unit segments and two legs of length $\sqrt{2}$ cut from corners). Subtract the four corner right triangles of leg $\sqrt{2}/\sqrt{2} = 1$:

$$\text{area} \;=\; (3 + 2\sqrt{2})^2 - 4 \cdot \tfrac{1}{2} \cdot (\sqrt{2})^2 \;=\; 13 + 12\sqrt{2}.$$



Problem 6.10

A triangle has sides a, b, c and circumradius R. Prove that the triangle is right-angled if and only if

$$a^2 + b^2 + c^2 \;=\; 8R^2.$$

BMO 2001

Solution. By the extended law of sines, $a = 2R \sin A$, etc., so $a^2 + b^2 + c^2 = 4R^2(\sin^2 A + \sin^2 B + \sin^2 C)$. The right-angle condition is that one of the angles, say C, is $\pi/2$, so $\sin^2 C = 1$ and $A + B = \pi/2$. Under the latter $\sin^2 A + \sin^2 B = \sin^2 A + \cos^2 A = 1$, giving the sum $= 2$ and $a^2 + b^2 + c^2 = 8R^2$.

Conversely, $\sin^2 A + \sin^2 B + \sin^2 C = 2$ in a triangle forces $\cos A \cos B \cos C = 0$, i.e. one angle is $\pi/2$.

Chapter 7

Combinatorics

This chapter collects enumeration and structural combinatorics problems. The underlying tools are familiar from any first course: bijection (Vandermonde's $\sum \binom{n}{k}^2 = \binom{2n}{n}$ falls out of choosing a committee), generating function (Catalan triangulations from the recurrence $T_n = \sum T_k T_{n-k+1}$), recurrence-and-solve (Stan Wagon's $2 \times n$ connected-set count produces the Pell sequence $3, 7, 17, 41, \ldots$, the numerators of the convergents of $\sqrt{2}$), inclusion–exclusion (the problème des ménages), and the pigeonhole principle. Two of the problems turn on a parity invariant that arises only when one notices that each variable appears in a specific number of summands.

Problem 7.1

In how many ways can a convex n-gon be triangulated by diagonals so that no two diagonals cross inside the polygon?

Solution. Label the vertices $1, 2, \ldots, n$ cyclically. Fix the edge from vertex 1 to vertex n and consider the triangle containing that edge; its third vertex is some k with $2 \le k \le n-1$, which partitions the remainder into a k-gon and an $(n-k+1)$-gon. The number of triangulations T_n therefore satisfies

$$T_n = \sum_{k=2}^{n-1} T_k \cdot T_{n-k+1},$$

which is the Catalan recurrence (with $T_2 = 1$). The closed form is $T_n = C_{n-2}$, where $C_m = \binom{2m}{m}/(m+1)$ is the m-th Catalan number.

Problem 7.2

How many ordered sequences $(a_1, a_2, \dots, a_n)$ of non-negative integers satisfy $a_1 + a_2 + \cdots + a_n = k$?

Solution. This is the classical "stars and bars" count. Arranging k stars and $n-1$ bars in a row, where each of the n sequence entries corresponds to the number of stars in one gap, gives $\binom{k+n-1}{n-1}$ arrangements.

Problem 7.3

How many strings of 1s and plus signs are there of length n, where by "length" we mean the total number of characters, and plus signs cannot appear at the start or end or be adjacent?

Solution. A valid string consists of some number of plus signs interspersed between groups of 1s. If there are k plus signs and hence $k+1$ groups of 1s (each of positive size), the total character count n equals k plus the sum of the group sizes. Fixing the group sizes with total $n-k$ and each part positive gives $\binom{n-k-1}{k}$ choices. Summing over $k \geq 0$:

$$\sum_{k\geq 0} \binom{n-k-1}{k} = F_n,$$

the n-th Fibonacci number.

Problem 7.4

n people attend a party and each shakes hands with a random subset of the others. Prove that at least two attendees shook hands with the same number of people.

Solution. The number of handshakes per person is in $\{0, 1, \dots, n-1\}$, but the values 0 and $n-1$ cannot both be attained (if someone has 0 handshakes, no one can have $n-1$). So the n people take values in a set of size at most $n-1$, and by pigeonhole two coincide.

PROBLEM 7.5

In how many ways can n couples be seated around a round table of $2n$ seats such that no couple sits adjacent?

Solution. This is the *problème des ménages*, solved by Touchard (1934) and given the explicit formula

$$M_n \;=\; 2\,n!\sum_{k=0}^{n}(-1)^k\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!.$$

The short derivation uses inclusion–exclusion over the set of "bad" events that a given couple is adjacent.

PROBLEM 7.6

How many subsets of $\{1,2,\dots,n\}$ contain no two consecutive integers?

Solution. Let $f(n)$ denote the count. Conditioning on whether n is in the subset: if yes, the subset of $\{1,\dots,n-2\}$ contributes $f(n-2)$; if no, the subset of $\{1,\dots,n-1\}$ contributes $f(n-1)$. With $f(0) = 1$ and $f(1) = 2$, the recurrence is Fibonacci-like: $f(n) = f(n-1) + f(n-2) = F_{n+2}$.

PROBLEM 7.7

Prove the Vandermonde identity

$$\sum_{k=0}^{n}\binom{n}{k}^2 \;=\; \binom{2n}{n}.$$

Solution. A committee of n must be chosen from a group of n men and n women. The right-hand side counts this directly. The left-hand side splits the count by how many men are chosen: if k men, then $\binom{n}{k}$ choices for them, and $\binom{n}{n-k} = \binom{n}{k}$ for the complementary $n-k$ women. Summing over k yields the identity.

PROBLEM 7.8

Consider a $2 \times n$ rectangle divided into $2n$ unit squares. How many connected subsets R of squares contain at least one square from the leftmost column and at least one from the rightmost column? (Connectedness is by shared edges.)

S. Wagon

Solution. Let $A(n)$ be the count; write $B(n) = A(n-1)$ for configurations whose rightmost column is fully occupied, and $L(n)$ for those whose rightmost column has exactly the top square (by symmetry, the bottom case doubles it). Then $A(n) = B(n) + 2L(n)$ and $L(n) = L(n-1) + A(n-2)$. Eliminating L yields the Pell-like recurrence

$$A(n) = 2A(n-1) + A(n-2), \qquad A(1) = 3,\ A(2) = 7.$$

The sequence $3, 7, 17, 41, 99, 239, 577, \ldots$ is precisely the numerators of the continued-fraction convergents of $\sqrt{2}$.

PROBLEM 7.9

Each of $a_1, a_2, \ldots, a_n$ is $+1$ or -1, and

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0.$$

Prove that $4 \mid n$.

Solution. Each summand is ± 1, so for $S = 0$ the number of $+1$'s must equal the number of -1's: n is even, and each count is $n/2$. The product of all n summands is $(-1)^{n/2}$. But that product is also

$$\prod_{i=1}^{n} a_i a_{i+1} a_{i+2} a_{i+3} = \left(\prod_{i=1}^{n} a_i\right)^4 = 1$$

(each a_i appears in exactly four summands). Hence $(-1)^{n/2} = 1$, so $n/2$ is even and $4 \mid n$.

Problem 7.10

In how many ways can $2n$ people be paired off into n teams of two?

BMO 1995

Solution. Choose the first pair in $\binom{2n}{2}$ ways, the next in $\binom{2n-2}{2}$ ways, and so on; the product is $(2n)!/2^n$. Since the n pairs are unordered, divide by $n!$:

$$\text{number of pairings} \;=\; \frac{(2n)!}{2^n\, n!}.$$

As a by-product, counting the same object as "choose n teams of m from mn people" in two ways yields the divisibility identity $\left((mn)!\right)^2 \mid (m!)^{n+1}(n!)^{m+1}$, which has no obvious prime-by-prime proof.

CHAPTER 8

Inequalities

INEQUALITIES FORM A RECURRING SUBJECT IN COMPETITION MATHEMATICS. A small library of named techniques covers a remarkable range of problems. AM-GM is the workhorse: applied to the right grouping of terms it produces tight bounds with explicit equality cases. Cauchy–Schwarz, especially in its Engel form $\sum a_i^2/b_i \ \geq\ (\sum a_i)^2/\sum b_i$, is the standard tool when the inequality involves fractions with quadratic numerators. Muirhead's inequality, founded on the notion of majorisation between exponent tuples, settles many symmetric inequalities at a stroke; three problems below are BMO questions of this flavour. The tangent-line trick (bounding a function by its tangent at the equality point) converts nonlinear sums to linear ones that the constraint then forces to zero. The chapter ends with two Muirhead applications and one six-term AM-GM split that yields a minimum of 96.

PROBLEM 8.1

Let a, b, c be positive reals with $a + b + c = 1$. Prove that

$$\left(1 + \tfrac{1}{a}\right)\left(1 + \tfrac{1}{b}\right)\left(1 + \tfrac{1}{c}\right) \ \geq\ 64.$$

Solution. Substituting $1 = a + b + c$ in each numerator gives $1 + 1/a = (2a + b + c)/a$, and similarly for the other two factors. The product becomes

$$\prod\left(1 + \tfrac{1}{a}\right) \ =\ \frac{(2a + b + c)(2b + a + c)(2c + a + b)}{abc}.$$

Apply AM-GM to each numerator: $2a+b+c = a+a+b+c \geq 4\sqrt[4]{a^2bc}$, etc. Multiplying the three:

$$\prod(2a+b+c) \;\geq\; 64\sqrt[4]{a^4b^4c^4} \;=\; 64\,abc.$$

Dividing by abc gives the inequality directly, with equality when $a = b = c = 1/3$.

Problem 8.2

Prove that for positive reals $x_1, x_2, \ldots, x_n$ summing to n,

$$\frac{1}{x_1^2+1} + \frac{1}{x_2^2+1} + \cdots + \frac{1}{x_n^2+1} \;\geq\; \tfrac{n}{2}.$$

Solution. The tangent line to $f(x) = 1/(x^2+1)$ at $x = 1$ is $y = 1/2 - (x-1)/2$. The function f lies above this tangent on $(0, \infty)$ because $f'' > 0$ there fails to hold uniformly but the bound holds directly: writing

$$\frac{1}{x^2+1} - \Big(\frac{1}{2} - \frac{x-1}{2}\Big) \;=\; \frac{(x-1)^2\,x}{2(x^2+1)} \;\geq\; 0 \qquad (x > 0).$$

Summing over i:

$$\sum_{i=1}^{n} \frac{1}{x_i^2+1} \;\geq\; \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{n}(x_i - 1) \;=\; \frac{n}{2} - \frac{1}{2}(n-n) \;=\; \frac{n}{2},$$

using $\sum x_i = n$. Equality at all $x_i = 1$.

Problem 8.3

Prove that for positive reals a, b, c,

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \;\geq\; \tfrac{3}{2}.$$

Solution. This is Nesbitt's inequality. Use Cauchy–Schwarz in Engel form:

$$\sum \frac{a}{b+c} \;=\; \sum \frac{a^2}{ab+ac} \;\geq\; \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

By AM-GM, $(a+b+c)^2 \geq 3(ab+bc+ca)$, so the sum is at least $3/2$. Equality at $a = b = c$.

Problem 8.4

Show that for $n \geq 1$,

$$\sum_{k=1}^{n} \frac{1}{k} \;>\; \ln(n+1).$$

Solution. Because $1/x$ is strictly decreasing,

$$\frac{1}{k} \;>\; \int_{k}^{k+1} \frac{dx}{x} \;=\; \ln(k+1) - \ln k.$$

Summing from $k = 1$ to n and telescoping gives $\sum 1/k > \ln(n+1)$.

Problem 8.5

Prove that among any three real numbers summing to 3, at least one is at most 1.

Solution. If all three exceed 1, their sum exceeds 3, contradicting the hypothesis. Hence one is at most 1.

Problem 8.6

Prove that for positive reals a, b,

$$4\,(a^3 + b^3) \;\geq\; (a+b)^3.$$

BMO 1996

Solution. Expanding the right-hand side,

$$4(a^3+b^3)-(a+b)^3 \;=\; 3a^3-3a^2b-3ab^2+3b^3 \;=\; 3(a-b)^2(a+b).$$

Since $a, b > 0$, the product $(a-b)^2(a+b) \geq 0$, with equality iff $a = b$. The inequality follows.

Problem 8.7

Positive reals x, y, z satisfy $xyz = 32$. Find the minimum value of $x^2 + 4xy + 4y^2 + 2z^2$.

BMO 2000 R2

Solution. Split the expression into six terms suited to AM-GM:

$$x^2 + 4xy + 4y^2 + 2z^2 \;=\; x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2.$$

By AM-GM on these six terms,

$$\frac{x^2 + 2xy + 2xy + 4y^2 + z^2 + z^2}{6} \;\geq\; \sqrt[6]{16\,(xyz)^4}.$$

With $xyz = 32 = 2^5$, the right-hand side equals $\sqrt[6]{2^{24}} = 16$, so the sum is at least 96. Equality holds at $x = 2y$ and $z^2 = 2xy$, i.e. at $(x, y, z) = (4, 2, 4)$.

Problem 8.8

Non-negative reals p, q, r satisfy $p + q + r = 1$. Prove that

$$7\,(pq + qr + rp) \;\leq\; 2 + 9\,pqr.$$

BMO 1999

Solution. Let $\Sigma = p^2q + p^2r + q^2p + q^2r + r^2p + r^2q$ denote the symmetric two-power sum. Cubing the constraint $p+q+r = 1$ gives

$$1 \;=\; (p^3 + q^3 + r^3) + 3\,\Sigma + 6\,pqr.$$

Doubling, then adding $9\,pqr$ to both sides,

$$2 + 9\,pqr \;=\; 2(p^3 + q^3 + r^3) + 6\,\Sigma + 21\,pqr.$$

On the other side, $7(pq + qr + rp)(p + q + r) = 7\,\Sigma + 21\,pqr$, so the inequality reduces to

$$2(p^3 + q^3 + r^3) \;\geq\; \Sigma.$$

This is Muirhead's inequality applied to the majorisation $(3, 0, 0) \succ (2, 1, 0)$.

Problem 8.9

Positive reals x, y, z satisfy $x^2 + y^2 + z^2 = 1$. Prove that

$$x^2yz + xy^2z + xyz^2 \;\leq\; \tfrac{1}{3}.$$

BMO 2002

Solution. Two applications of Muirhead's inequality (justified by the majorisations $(4,0,0) \succ (2,1,1)$ and $(2,2,0) \succ (2,1,1)$):

$$2(x^2yz + xy^2z + xyz^2) \;\le\; 2(x^4 + y^4 + z^4),$$

$$2(x^2yz + xy^2z + xyz^2) \;\le\; 2(x^2y^2 + x^2z^2 + y^2z^2).$$

Adding,

$$3(x^2yz + xy^2z + xyz^2) \;\le\; (x^2 + y^2 + z^2)^2 \;=\; 1.$$

Equality at $x = y = z = 1/\sqrt{3}$.

Index of Techniques

The book is organised by problem type, but the underlying techniques cut across chapters. A reader looking for the canonical use of a particular method will find the relevant problems gathered below. Where two techniques compete for billing the problem appears under both.

Generating functions

- Fibonacci-ratio identity, $\sum F_n/a^{n+1} = \sum F_{2n}/(a+1)^{n+1}$ (*Problem 1.1*)
- Quarter-period exponential, $\sum((-1)^{\lceil n/2 \rceil} - (-1)^n)/n!$ (*Problem 1.2*)
- Raghava's $\pi(\pi+2)/2$ sum (*Problem 1.3*)
- $\sum n/2^n$ via differentiation (*Problem 1.5*)
- $\sum H_n/(n \cdot 2^n) = \pi^2/12$ via dilogarithm (*Problem 1.8*)
- Lucas-number sum $\sum L_n/10^n = 12/89$ (*Problem 1.11*)

Telescoping

- $\sum \arctan(2/(2n+1)^2) = \pi/2$ (*Problem 1.4*)
- $\sum 1/(n^2-1) = 3/4$ (*Problem 1.6*)
- $f(1) = 1996$, $\sum f = n^2 f(n)$, find $f(1996)$ (*Problem 4.8*)
- Gauss totient sum $\sum_{d|360} \varphi(d) = 360$ (*Problem 5.5*)

Order swap and double sums

- Raghava's $\sum\sum 1/(k^n \cdot k!) = 3 - e$ (*Problem 1.9*)
- Stanley's $\pi^4/90$ via 2-adic split of $\zeta(2)^2$ (*Problem 1.10*)

PARAMETRIC DIFFERENTIATION (FEYNMAN TRICK)

- $\int_0^{\pi/2} x/\tan x\, dx = \pi \ln 2/2$ (*Problem 2.1*)
- $\int_0^\infty (e^{-ax} - e^{-bx})/x\, dx$ (Frullani) (*Problem 2.4*)
- $\int_0^\infty \sin x/x\, dx = \pi/2$ (*Problem 2.5*)

SUBSTITUTION AND SYMMETRY IN INTEGRALS

- Weierstrass substitution: $\int (1 + \cos\theta \cos x)^{-1}\, dx = \theta/\sin\theta$ (*Problem 2.9*)
- Reciprocal substitution: $\int 1/(1 + x^4)\, dx = \pi/(2\sqrt{2})$ (*Problem 2.8*)
- $\int \ln \sin x = -\pi \ln 2/2$ via reflection (*Problem 2.2*)
- Polar Gaussian $\int e^{-x^2}$ (*Problem 2.3*)

SERIES-THEN-INTEGRATE

- $\int \ln(1 + x)/x\, dx = \pi^2/12$ (*Problem 2.6*)
- Stanley's iterated $\int\int = \pi^2/12 - \ln 4 + 1$ (*Problem 2.10*)
- $\int x/(e^x - 1)\, dx = \pi^2/6$ (*Problem 2.7*)

INTEGRAL EQUATIONS AND ODES (PROBABILITY)

- Camels and straws, $\mathbb{E}[N] = e$ (*Problem 3.1*)
- Last-straw weight, $\mathbb{E}[X] = 2 - e/2$ (*Problem 3.3*)
- Adding pairs of uniforms, $\mathbb{E}[N] = e(\sin 1 + \cos 1)$ (*Problem 3.4*)
- Stanley's quarter-disc, $\mathbb{E}[N] = e^{\pi/4}$ (*Problem 3.11*)

INDICATOR LINEARITY AND SYMMETRY

- Fixed points of a permutation (*Problem 3.5*)
- Birds on a wire (*Problem 3.7*)
- Cards to first ace (*Problem 3.12*)
- Quarter-circle event $\pi/4 - 1/2$ (*Problem 3.13*)
- Random walk on C_n cover time (*Problem 3.10*)

FIRST-STEP CONDITIONING

- Ant on a cube (*Problem 3.8*)
- String cutting, $\mathbb{E}[Y] = 1 - \pi^2/12$ (*Problem 3.6*)

SYMMETRIC POLYNOMIALS AND NEWTON'S IDENTITIES

- $x + y = 3xy, x^2 + y^2 = 1/3$ (*Problem 4.1*)
- $x^3 + y^3 + (x + y)^3 + 30xy = 2000$ (*Problem 4.1*)
- $x^2 + y^2 = 31, x^3 + y^3 = 154$, max $\Re(x + y)$ (*Problem 4.2*)

CONJUGATE SUBSTITUTION AND RECIPROCALS

- $(4 + \sqrt{15})^x + (4 - \sqrt{15})^x = 62$ (*Problem 4.5*)
- Nested radical $\sqrt{2 + \sqrt{2 + \cdots}} = 2$ (*Problem 4.7*)
- $x^2 + x^2/(x + 1)^2 = 3$ via $y + 1/y$ (*Problem 4.7*)

CYCLIC FUNCTIONAL EQUATIONS

- $f(x) + f(-1/(x - 1)) = \cdots$ (*Problem 4.4*)
- Ramanujan's $x^y = y^x$ in rationals (*Problem 4.6*)

MODULAR AND DIVISIBILITY TRICKS

- BMO 1993, $2^{m-1} \equiv 1 \pmod{m}$ via FLT (*Problem 5.1*)
- Sophie Germain: $n^4 + 4$ never prime for $n > 1$ (*Problem 5.3*)

- $30 \mid n^5 - n$ (*Problem 5.4*)
- $2^p + p^2$ prime $\Rightarrow p = 3$ (*Problem 5.6*)
- Repunits $11, 111, \ldots$ never squares (mod 4) (*Problem 5.8*)

PELL EQUATIONS

- BMO 1994, average of squares first square (*Problem 5.2*)
- Wagon's $2 \times n$ connected sets (*Problem 7.8*)

SUM-OF-CUBES AND ALGEBRAIC IDENTITIES

- $m^3 + n^3 + 99mn = 33^3$ (*Problem 5.7*)

REFLECTION AND SYMMETRY (GEOMETRY)

- $\angle ADE - \angle CDB = 30^\circ$ via reflection (*Problem 6.3*)
- Wagon's polygon-projection sum $= n$ (*Problem 6.6*)

LOCUS AND POWER OF A POINT

- Dorofeev's tangency locus = semicircle (*Problem 6.4*)
- Stanley's points closer to a diameter, area $4/3$ (*Problem 6.5*)

DISSECTION AND AREA-PRESERVING TRANSFORMATION

- Mabry's inner hexagon, $1/13$ of the outer (*Problem 6.7*)
- Inscribed octagon, area $13 + 12\sqrt{2}$ (*Problem 6.9*)

TRIGONOMETRIC IDENTITIES AND THE LAW OF SINES

- BMO 2001, right triangle iff $a^2 + b^2 + c^2 = 8R^2$ (*Problem 6.10*)

APOLLONIUS AND INCIRCLE/CIRCUMCIRCLE

- Right-triangle area = 228 from $\overline{OI} = 8$ (*Problem 6.8*)

RECURRENCE AND BIJECTION (COMBINATORICS)

- Catalan triangulations (*Problem 7.1*)
- Stars and bars (*Problem 7.2*)
- Fibonacci subset count (*Problems 7.6, 7.7*)
- 1s and +s string count $= F_n$ (*Problem 7.3*)
- Vandermonde's identity (*Problem 7.7*)

PIGEONHOLE AND PARITY

- Handshake counts coincide (*Problem 7.4*)
- ± 1 cyclic 4-product sum $= 0 \Rightarrow 4 \mid n$ (*Problem 7.9*)
- Three-real with sum 3 has one ≤ 1 (*Problem 8.5*)

INCLUSION–EXCLUSION AND COUNTING

- Problème des ménages (*Problem 7.5*)
- BMO 1995, $(2n)!/(2^n n!)$ pairings (*Problem 7.10*)

AM-GM, GM-HM, AND THE TANGENT-LINE TRICK

- $(1 + 1/a)(1 + 1/b)(1 + 1/c) \geq 64$ (*Problem 8.1*)
- Tangent-line trick on $1/(x^2 + 1)$ (*Problem 8.2*)
- $4(a^3 + b^3) \geq (a + b)^3$ (*Problem 8.6*)
- BMO 2000 R2 min = 96 via 6-term AM-GM split (*Problem 8.7*)

CAUCHY–SCHWARZ

- Nesbitt $\sum a/(b + c) \geq 3/2$ in Engel form (*Problem 8.3*)

- Cauchy–Schwarz on Lagrange residual, $\max f = (3 + \sqrt{15})/2$ (*Problem 4.9*)

MUIRHEAD'S INEQUALITY

- BMO 1999, $7(pq + qr + rp) \leq 2 + 9pqr$ (*Problem 8.8*)
- BMO 2002, $xyz(x + y + z) \leq 1/3$ (*Problem 8.9*)

INTEGRAL COMPARISON

- $\sum 1/k > \ln(n + 1)$ (*Problem 8.4*)