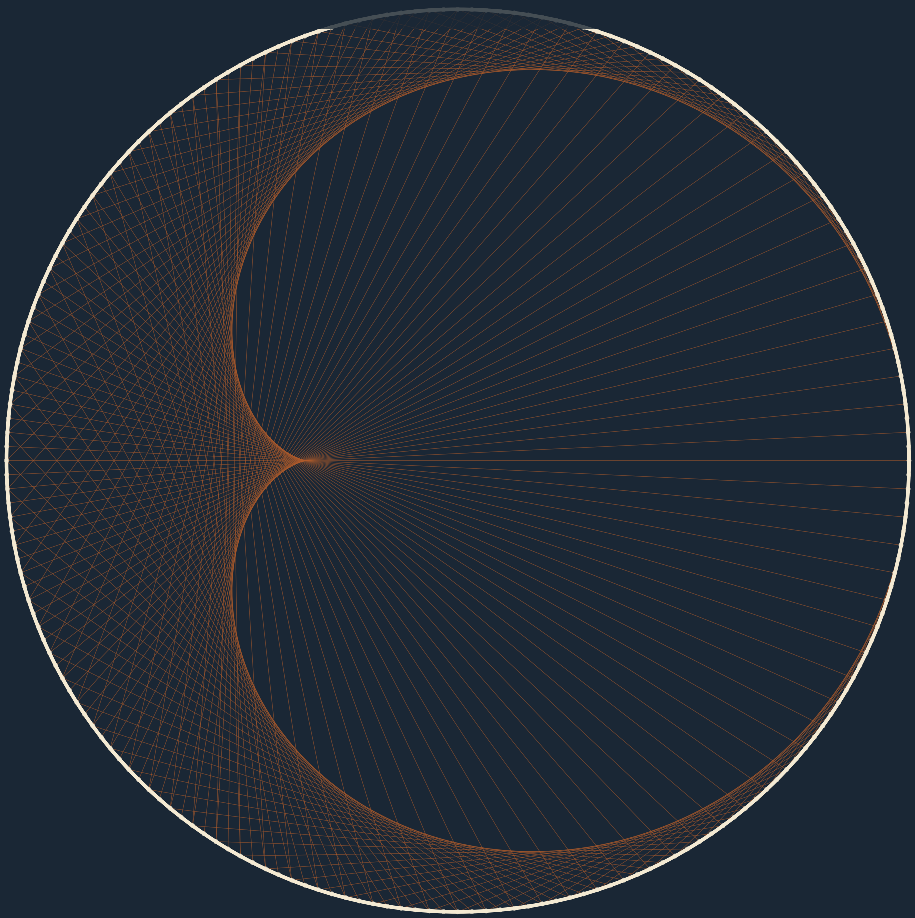
Curated Mathematics

Number Puzzles

Fifty classic puzzles, with worked solutions



Vamshi Jandhyala · London

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Fifty classical number puzzles, recast in contemporary language and given fresh, fully worked solutions.

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A Note on These Puzzles

This is a small collection of classic number puzzles, the kind that have circulated for a century or more in magazines, common rooms and the backs of notebooks. Each has been chosen for one reason: that it hides a genuinely pretty piece of reasoning, and rewards you the moment you see it.

Every puzzle is its own short chapter. The statement comes first, set in plain modern language, and the solution follows at once, worked through in full rather than merely announced. The aim throughout is the shortest honest argument, the one that makes you nod rather than the one that grinds to the answer. Where a puzzle turns on a single idea, casting out nines, a difference of two squares, counting in threes, the idea is named and explained, so that the same trick is yours to use again.

Nothing here asks for mathematics beyond school. A reader comfortable with ordinary arithmetic, a little algebra and the patience to follow a clear chain of reasoning will be able to read every solution to the end. A few puzzles brush against deep results, and where a fact has been settled by mathematics heavier than these pages can carry, this is said plainly and the result is taken on trust rather than dressed up as obvious.

The puzzles can be read in any order. Cover the solution, attempt the puzzle, and consult the working only when you are ready. Half the pleasure is in the attempt, and the other half in seeing how little, in the end, the answer really needed.

Contents

Puzzle 1

The Suppressed Digit

Here is a trick to play on a friend. Ask them to do the following out of your sight.

1. Write down any whole number with more than five digits.
2. Add up its digits, and subtract that total from the number itself.
3. From the result, cross out any one digit that is not a zero.
4. Read out the digits that remain, in any order they please.

You then name the digit they crossed out. If they read out $2, 2, 3, 5, 9$, you would tell them at once that the suppressed digit was a 6. How is it done?

Solution

The trick rests on a single fact about the number nine: a whole number and the sum of its digits leave the same remainder on division by nine. This holds because 10, 100, 1000 and so on are each one more than a multiple of nine, so every digit contributes only itself to the remainder.

It follows that subtracting the digit sum from the number always leaves an exact multiple of nine. And the digits of a multiple of nine themselves add up to a multiple of nine.

So the digits of the result, all of them together, add up to a multiple of nine. The ones read out to you sum to $2 + 2 + 3 + 5 + 9 = 21$. The next multiple of nine above 21 is 27, so the suppressed digit must be $27 - 21 = 6$.

This also explains the rule against crossing out a zero. If the digits you hear already add up to a multiple of nine, the hidden digit is a 9, not a 0, since a suppressed 0 and a suppressed 9 could not be told apart. Barring zero removes the ambiguity.

Puzzle 2

Brandy and Wine

✠

A merchant has six casks, holding 8, 13, 15, 17, 19 and 31 litres. He stocks only two drinks, brandy and wine, and his brandy costs exactly twice as much per litre as his wine.

Two customers arrive. The first buys only brandy and spends exactly £28. The second buys only wine and also spends exactly £28. No cask is ever broken into: each is sold whole or not at all. When the customers leave, a single cask remains unsold.

Taking that last cask to hold brandy, what is it worth?

Solution

Both customers spend the same £28, but brandy costs twice as much per litre as wine, so the wine buyer must have carried off exactly twice the volume of the brandy buyer.

The five casks that were sold therefore split into a brandy share and a wine share twice as large. Together they come to three times the brandy volume, so the volume sold is a multiple of three.

All six casks hold $8 + 13 + 15 + 17 + 19 + 31 = 103$ litres. For the sold volume to be a multiple of three, the unsold cask must leave a multiple of three behind. Looking at remainders on division by three, only the 13, 19 and 31 litre casks qualify.

- Leave the 13 litre cask: 90 litres are sold, 30 of brandy and 60 of wine. No selection of $8, 15, 17, 19, 31$ adds up to 30, so this fails.

- Leave the 31 litre cask: 72 litres are sold, 24 of brandy and 48 of wine. No selection of $8, 13, 15, 17, 19$ adds up to 24, so this fails as well.

- Leave the 19 litre cask: 84 litres are sold, 28 of brandy and 56 of wine. This works: the brandy is $13 + 15 = 28$ litres, and the wine is $8 + 17 + 31 = 56$ litres.

So the wine sells at 50 pence a litre and the brandy at £1 a litre. The wine buyer pays $56 \times 0.5 = 28$ pounds and the brandy buyer $28 \times 1 = 28$ pounds, exactly as required. The unsold cask is the 19 litre one, and as brandy it is worth 19 pounds, that is £19.

Puzzle 3

The Missing Term

A student was copying out a sequence for his homework when the telephone rang. While he was gone, his younger brother rubbed out one of the terms. Here is what was left:

10, 11, 12, 13, 14, 15, 16, 17, 20, 22, 24, ___, 100, 121, 10000.

What was the number that was rubbed out?

Solution

Every term is the number sixteen, written in a different base, with the bases counting down from sixteen to two.

In base sixteen, sixteen is written 10; in base fifteen, 11; in base fourteen, 12; and so on down to base ten, where it is the familiar 16, then base nine (17), and base eight (20, since $16 = 2 \times 8$), base seven (22), base six (24). The next base down is five, and $16 = 3 \times 5 + 1$, so there sixteen is written 31. The list then finishes with base four (100, since $16 = 4^2$), base three (121, since $16 = 9 + 6 + 1$) and base two (10000).

The missing term is 31, which is sixteen written in base five.

Puzzle 4

The Club of Six

Six friends, the men Tom, Dick and Harry and the women Anna, Cathy and Lucy, make up three married couples. Which man is married to which woman is exactly what we are asked to find.

One day they each went to market and bought some sheep. By chance, each of them bought as many sheep as the number of pounds they paid for a single one, so a person who bought n sheep paid n pounds apiece and spent n^2 pounds in all.

Comparing receipts afterwards, they noticed two things. Each husband had spent exactly £63 more than his own wife. And Tom had bought 23 more sheep than Cathy, while Dick had bought 11 more than Lucy.

Who is married to whom?

Solution

If a husband bought h sheep and his wife w, he spent h^2 pounds and she spent w^2, and we are told $h^2 - w^2 = 63$. Written as $(h - w)(h + w) = 63$, and since 63 is odd both factors must be odd. The three ways to split 63 into two odd factors,

$$(h - w,\ h + w) = (1, 63), \quad (3, 21), \quad (7, 9),$$

give $(h, w) = (32, 31)$, $(12, 9)$ and $(8, 1)$. So the husbands bought 32, 12 and 8 sheep, and the wives bought 31, 9 and 1.

Now use the clues. Tom bought 23 more than Cathy. Among the husbands' counts $32, 12, 8$ and the wives' counts $31, 9, 1$, the only difference equal to 23 is $32 - 9$, so Tom bought 32 and Cathy bought 9. Dick bought 11 more than Lucy; of the men left, 12 and 8, and the women left, 31 and 1, the only difference equal to 11 is $12 - 1$, so Dick bought 12 and Lucy bought 1. That leaves Harry with 8 and Anna with 31.

Finally, pair each husband with his wife using the counts found at the start: the man who bought 32 is married to the wife who bought 31, the one who bought 12 to the wife who bought 9, and the one who bought 8 to the wife who bought 1. Therefore

Tom is married to Anna, Dick to Cathy, Harry to Lucy.

The pleasing twist is that neither woman named in the clues is the wife of the man she is compared with.

Puzzle 5

One Hundred, and One

1. Take the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$ in that order. Placing only plus and minus signs between them, and running neighbours together into larger numbers where you wish, make the result equal to 100.
2. Using all ten digits 0 through 9, each exactly once, write an expression equal to 1.

Solution

Part 1. A tidy answer using only three signs is

$$123 - 45 - 67 + 89 = 100.$$

Reading from the left, $123 - 45 = 78$, then $78 - 67 = 11$, then $11 + 89 = 100$. There are others, for instance $1 + 2 + 3 - 4 + 5 + 6 + 78 + 9 = 100$, but the first is the neatest.

Part 2. Make two halves and add them:

$$\frac{148}{296} + \frac{35}{70} = \frac{1}{2} + \frac{1}{2} = 1.$$

Each fraction is exactly one half, since $296 = 2 \times 148$ and $70 = 2 \times 35$, and between the two of them the ten digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ each appear once.

PUZZLE 6

A Stroke of Genius

The English mathematician G. H. Hardy once visited his collaborator Srinivasa Ramanujan, who was ill in hospital. For want of an opening remark, Hardy said that he had come in taxicab number 1729, and that it struck him as a rather dull number. Ramanujan replied at once that, on the contrary, it was a most interesting one: it was the smallest number expressible as the sum of two cubes in two different ways.

Complete Ramanujan's sentence with four words, two of them the same, that add up to four: the smallest number that is the sum of two ______ in ______ ______.

Solution

The missing words are *cubes*, in *two ways*: *two* cubes in *two* ways, and indeed $2 + 2 = 4$. The two ways are

$$1729 = 1^3 + 12^3 = 9^3 + 10^3,$$

since $1 + 1728 = 1729$ and $729 + 1000 = 1729$. No smaller number is a sum of two positive cubes in two distinct ways, so 1729 is the smallest. In Ramanujan's honour, numbers with this kind of property are now called *taxicab numbers*.

Puzzle 7

Two Games of Chance

Two friends spend a wet afternoon with a pair of dice.

1. In the first game each rolls a single die. What is the chance that one named player rolls strictly higher than the other?
2. In the second they roll both dice and add the scores. Which total should they bet on?
3. In the third they put the dice away and instead draw a rectangle, choosing each of its two side lengths at random somewhere between 0 and 1 inch. What is the chance that the rectangle's diagonal is shorter than an inch?

Solution

Part 1. There are 36 equally likely pairs of rolls. The two players tie in 6 of them, a chance of one in six. By symmetry the remaining 30 split evenly between 'the first rolls higher' and 'the second rolls higher', so each has probability $15/36 = 5/12$.

Part 2. The total ranges from 2 to 12. Counting the ways to make each, the total 7 arises most often, in 6 of the 36 cases (again one in six), more than any other total; 6 and 8 come next, with five ways each. So bet on 7.

Part 3. Picture the two side lengths as a single point chosen uniformly in the unit square, its coordinates the two lengths.

The diagonal $\sqrt{x^2 + y^2}$ is shorter than 1 exactly when the point lies inside the quarter circle of radius 1 drawn from the corner. The chance is the quarter circle's area divided by the square's area,

$$\frac{\pi/4}{1} = \frac{\pi}{4} \approx 0.79,$$

a little under four times in five.

Puzzle 8

Dots in Triangles and Pyramids

1. The triangular numbers count dots in a triangular array, $1, 3, 6, 10, \ldots$, the nth being $1 + 2 + \cdots + n$. If a triangle is built from 2701 dots, how many more dots are needed for the next larger triangle?

2. The numbers 1 and 36 are each at once a triangular number and a perfect square. What are the next four numbers that are both?

3. Stacking triangles gives the tetrahedral numbers $1, 4, 10, 20, \ldots$, the nth being the sum of the first n triangular numbers. Show that two consecutive tetrahedral numbers add to a square pyramidal number (the count of cannonballs in a square-based pyramid, $1^2 + 2^2 + \cdots + n^2$). Then find every square pyramidal number, other than 1, that is also a perfect square.

Solution

Part 1. The nth triangular number is $n(n+1)/2$. Setting this to 2701 gives $n(n+1) = 5402$, so $n = 73$ (since $73 \times 74 = 5402$).

The next triangle simply adds one more row, of 74 dots. So 74 more are needed.

Part 2. The next four square-triangular numbers after 1 and 36 are

$$1225 = 35^2, \quad 41616 = 204^2,$$
$$1413721 = 1189^2, \quad 48024900 = 6930^2,$$

and each is also triangular $(T_{49}, T_{288}, T_{1681}, T_{9800})$. They obey the tidy rule $a_{n+1} = 34a_n - a_{n-1} + 2$: for instance $34 \times 36 - 1 + 2 = 1225$ and $34 \times 1225 - 36 + 2 = 41616$.

Part 3. Writing the tetrahedral number as $Te_n = n(n+1)(n+2)/6$,

$$\begin{aligned} Te_n + Te_{n-1} &= \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6} \\ &= \frac{n(n+1)[(n+2)+(n-1)]}{6} = \frac{n(n+1)(2n+1)}{6}, \end{aligned}$$

which is exactly $1^2 + 2^2 + \cdots + n^2$, the square pyramidal number. As for which of these are perfect squares: apart from the trivial 1, the only one is

$$4900 = 70^2 = 1^2 + 2^2 + \cdots + 24^2.$$

This is the famous cannonball problem, that a square layer of 70×70 holds exactly as many as a square pyramid 24 high. That 4900 is the sole non-trivial solution was proved by G. N. Watson in 1918, so we take it as settled rather than reprove it here.

Puzzle 9

Jack and Jill

Jack and Jill leave home at noon and walk along a level road, then up a hill, then back down the same hill and home again, without stopping. They walk at 4 miles an hour on the level, 3 uphill and 6 downhill, and they reach home at six o'clock. At what time did they reach the top of the hill? Give the answer to within half an hour.

Solution

Let the level stretch be a miles each way and the hill b miles each way. Going out they cover a on the level at 4 and b uphill at 3; coming back, b downhill at 6 and a on the level at 4. The whole journey takes

$$\frac{a}{4} + \frac{b}{3} + \frac{b}{6} + \frac{a}{4} = \frac{a}{2} + \frac{b}{2} = \frac{a+b}{2}$$

hours. This equals six, so $a + b = 12$.

The time to reach the top is the outward part alone,

$$t = \frac{a}{4} + \frac{b}{3} = \frac{12-b}{4} + \frac{b}{3} = 3 + \frac{b}{12}.$$

Since b lies between 0 and 12, the time t lies between 3 and 4 hours. So they reached the top between three and four o'clock, and the best single answer, good to within half an hour, is half past three.

Puzzle 10

Square Palindromes

A palindrome reads the same forwards and backwards. Some perfect squares are palindromes:

$$1^2 = 1, \qquad 11^2 = 121, \qquad 111^2 = 12321.$$

Why does this pattern hold, how far does it run, and what other palindromic squares are there?

Solution

The pattern belongs to the repunits, the numbers made only of ones. As long as no column total reaches ten, squaring a string of k ones makes the digits climb $1, 2, \ldots, k$ and back down:

$$1111^2 = 1234321, \qquad 11111^2 = 123454321,$$

and so on up to

$$111111111^2 = 12345678987654321 \quad \text{(nine ones)}.$$

The reason is that a string of k ones squared is the sum of k shifted copies of that string, and the jth column simply counts how many copies overlap there, a count that rises to k in the middle and falls away symmetrically. With ten ones or more, a column total reaches ten, carrying begins, and the neat palindrome is lost: $1111111111^2 = 1234567900987654321$, which is not a palindrome.

Palindromic squares need not come from palindromic roots. Among the smaller ones,

$$22^2 = 484, \quad 26^2 = 676, \quad 212^2 = 44944,$$
$$264^2 = 69696, \quad 836^2 = 698896,$$

and of these roots only 22 and 212 are themselves palindromes.

Puzzle 11

Four Consecutive Numbers

✠

Find a triangle whose three sides and one of its heights are four consecutive whole numbers.

Solution

Take the triangle with sides 13, 14, 15, and drop the height onto the side of length 14. It meets that side at the point splitting it into lengths 5 and 9. Since

$$5^2 + 12^2 = 13^2 \qquad \text{and} \qquad 9^2 + 12^2 = 15^2,$$

the two pieces are the bases of right triangles of height 12 whose slanting sides are 13 and 15. Glued along their shared side of length 12, they form a triangle with base $5 + 9 = 14$, other sides 13 and 15, and height 12. So the three sides and the height are $12, 13, 14, 15$, four consecutive whole numbers.

(Equivalently, Heron's formula gives the area of the $13, 14, 15$ triangle as 84, so the height onto the side of length 14 is $2 \times 84/14 = 12$.)

Puzzle 12

Divisibility by Seventy-Three

Can you tell whether a large number is divisible by 73 without actually dividing? You may use the fact that $73 \times 137 = 10001$.

Solution

The key is that 10000 is one less than 10001, so 10000 leaves remainder -1 on division by 10001. Break the number into blocks of four digits, counting from the right, and form the alternating sum of those blocks. The number and that alternating sum leave the same remainder on division by 10001, and so on division by 73 as well (and by 137), since both divide 10001.

For example, take 84008765. Its four-digit blocks from the right are 8765 and 8400, and

$$8765 - 8400 = 365 = 73 \times 5.$$

The alternating sum is a multiple of 73, so the original number is too, and indeed $84008765 = 73 \times 1150805$. The same blocks show it is *not* divisible by 137, since 365 is not.

Puzzle 13

Pick Five, Always Sixty-Five

From the grid below, pick any number and cross out its row and its column. From what is left, pick another and again cross out its row and column. Carry on until you have chosen five numbers, one from each row and one from each column. Add them up.

1	6	11	16	21
2	7	12	17	22
3	8	13	18	23
4	9	14	19	24
5	10	15	20	25

You always reach 65, whichever five you took. Why? And how many different selections are there?

Solution

Write the labels $0, 1, 2, 3, 4$ down the left of the grid and $1, 6, 11, 16, 21$ across the top. Then the entry in any cell is exactly the sum of its row label and its column label: row 3 (label 2), column 4 (label 16) holds $2 + 16 = 18$, and so on throughout.

A valid selection takes one entry from each row and one from each column, so among the five chosen entries every row label appears once and every column label appears once. The total is therefore

$$(0 + 1 + 2 + 3 + 4) + (1 + 6 + 11 + 16 + 21) = 10 + 55 = 65,$$

whatever cells were picked.

As for how many selections there are: choosing one column for each row, with no column used twice, is the same as arranging the five columns in order, so there are $5! = 120$ of them. Any grid built as such an addition table of row and column labels has the same charm, and the constant is simply the total of all the labels.

Puzzle 14

Cancelling, and a Cube Trick

1. A careless student reduces $\frac{16}{64}$ to $\frac{1}{4}$ by "cancelling the sixes", and is somehow right. He does the same to $\frac{26}{65}$ and $\frac{19}{95}$, again correctly. Are there other two-digit fractions, not merely multiples of these, where striking out a shared digit gives the true value?

2. The same student takes cube roots by adding digits: the cube root of 512 is $5+1+2 = 8$, and of 4913 is $4+9+1+3 = 17$, both correct. Find every number, beyond the trivial 1, that equals the cube of its own digit sum.

Solution

Part 1. Apart from trivialities like $\frac{11}{11}$, there are exactly four such two-digit fractions:

$$\frac{16}{64} = \frac{1}{4}, \qquad \frac{19}{95} = \frac{1}{5}, \qquad \frac{26}{65} = \frac{2}{5}, \qquad \frac{49}{98} = \frac{1}{2},$$

where striking out the repeated digit (the 6, 9, 6, 9 in turn) leaves the correct reduced fraction.

To see there are no others, write such a fraction as $\frac{10a+b}{10b+c}$, where b is the shared digit (the units of the top, the tens of the bottom). Wanting the careless cancellation $\frac{10a+b}{10b+c} = \frac{a}{c}$ to be true, cross-multiply:

$$c(10a+b) = a(10b+c), \qquad \text{that is} \qquad 9ac = b(10a-c).$$

Here a, b, c are digits from 1 to 9. Running through the few possibilities, this equation holds in exactly four cases, the (a, b, c) being $(1, 6, 4)$, $(1, 9, 5)$, $(2, 6, 5)$ and $(4, 9, 8)$, which are precisely the four fractions above. So the schoolboy's luck runs out after these.

Part 2. We want numbers equal to the cube of their digit sum. Running through the cubes and checking, the complete list, beyond $1^3 = 1$, is

$$8^3 = 512, \quad 17^3 = 4913, \quad 18^3 = 5832,$$
$$26^3 = 17576, \quad 27^3 = 19683,$$

whose digit sums are $8, 17, 18, 26, 27$ respectively. There are no others: a number with d digits has digit sum at most $9d$, which grows far more slowly than its own cube root, so beyond a point a number always overtakes the cube of its digit sum.

Puzzle 15

A Pandigital Divisible by Eleven

✠

Write a nine-digit number using each of the digits 1 to 9 exactly once. What is the probability that it is divisible by 11? A club of twenty-three once each wrote such a number, found two of them divisible by 11, and declared the probability to be $\frac{2}{23}$. Were they right?

Solution

A number is divisible by 11 when the sum of its digits in the odd positions and the sum in the even positions differ by a multiple of 11. A nine-digit number has five odd positions and four even ones. Let E be the sum of the four even-position digits; since all nine digits total 45, the odd-position digits sum to $45 - E$. The test asks that

$$(45 - E) - E = 45 - 2E$$

be a multiple of 11, which happens exactly when E leaves remainder 6 on division by 11.

The four even-position digits are distinct digits from 1 to 9, so E lies between $1 + 2 + 3 + 4 = 10$ and $6 + 7 + 8 + 9 = 30$. The only values in that range leaving remainder 6 are $E = 17$ and $E = 28$. Among the four-digit subsets of $\{1, \dots, 9\}$, nine sum to 17 and two sum to 28 (these two being $\{4, 7, 8, 9\}$ and $\{5, 6, 8, 9\}$), eleven choices in all.

For each such choice of which digits sit in the even positions, there are 4! ways to arrange them and 5! ways to arrange the other five. So out of all 9! arrangements, the proportion divisible by 11 is

$$\frac{11 \times 4! \times 5!}{9!} = \frac{11}{126} \approx 0.087.$$

The club's $\frac{2}{23} \approx 0.087$ was a sample frequency that landed close to the truth by luck. The exact probability is $\frac{11}{126}$.

Puzzle 16

Casting Out in Threes

✠

Here is a quick test for divisibility by 37, resting on the fact that $37 \times 27 = 999$. Use it to decide whether 73926 is divisible by 37.

Solution

Because 999 is a multiple of 37, the number 1000 leaves remainder 1 on division by 37. So a number leaves the same remainder as the sum of its digits taken in three-digit groups from the right. For 73926 the groups are 926 and 073, and

$$926 + 73 = 999 = 27 \times 37,$$

a multiple of 37 (and of 27), so 73926 is divisible by both. Indeed $73926 = 37 \times 1998$.

The companion fact $1001 = 7 \times 11 \times 13$ gives a matching rule: since 1000 leaves remainder -1 on division by 1001, a number leaves the same remainder as the *alternating* sum of its three-digit groups, on division by 7, 11 and 13 all at once.

Puzzle 17

A Prime Number of Rails

✠

1. A right-angled triangular field has one side exactly 47 rails long, and all three sides are a whole number of rails. How many rails are needed to fence it right round?

2. What if that side were 48 rails instead?

Solution

Part 1. We need a right triangle with whole-number sides, one of them 47. As 47 is far too short to be the hypotenuse, take it as a leg, with other leg b and hypotenuse c. Then

$$47^2 = c^2 - b^2 = (c-b)(c+b).$$

Since 47 is prime, $47^2 = 2209$ can only be written as 1×2209, so $c - b = 1$ and $c + b = 2209$, giving $c = 1105$ and $b = 1104$. The sides are 47, 1104, 1105 (and indeed $47^2 + 1104^2 = 1105^2$), so the fence needs

$$47 + 1104 + 1105 = 2256 \text{ rails.}$$

A prime leg pins the triangle down completely: there is only one such field.

Part 2. With 48 the trick fails, because $48^2 = 2304$ splits in many ways into two factors of the same parity, and each gives a different triangle, for example

$$(36, 48, 60), \quad (20, 48, 52), \quad (48, 55, 73),$$
$$(48, 64, 80), \quad (48, 90, 102), \quad \ldots, \quad (48, 575, 577).$$

So the question now has no single answer; the field could be fenced in many different ways. A prime side fixes a right triangle, a composite side does not.

Puzzle 18

Nine Digits, the Unit Fractions

Using each of the nine digits 1 to 9 exactly once, some in the numerator and the rest in the denominator, write a fraction equal to $\frac{1}{2}$. Then do the same for $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$ and $\frac{1}{9}$.

Solution

One solution for each, though there are others:

$$\frac{1}{2} = \frac{6729}{13458}, \quad \frac{1}{3} = \frac{5832}{17496}, \quad \frac{1}{4} = \frac{3942}{15768},$$
$$\frac{1}{5} = \frac{2697}{13485}, \quad \frac{1}{6} = \frac{2943}{17658}, \quad \frac{1}{7} = \frac{2394}{16758},$$
$$\frac{1}{8} = \frac{3187}{25496}, \quad \frac{1}{9} = \frac{6381}{57429}.$$

In each, the four-digit numerator times the whole number on the left gives the five-digit denominator, and between numerator and denominator the digits 1 to 9 each appear once. For instance $6729 \times 2 = 13458$ and $6381 \times 9 = 57429$.

Puzzle 19

A Father's Fair Division

A man owned 36 houses, numbered 1 to 36. The rent from each house, per month, was £100 times its number: house 1 brought in £100, house 2 brought in £200, and so on up to house 36 at £3600. He had six sons, and wished to leave each son exactly six houses so that all six sons drew equal rent. How should he divide them? And if instead he had owned 100 houses, numbered alike, to leave equally among ten sons, ten houses each?

Solution

The rents are all £100 times a house number, so equal rent means equal totals of house numbers. The numbers 1 to 36 add to $\frac{36\times37}{2} = 666$, which split six ways is 111 each. So each son's six houses must have numbers summing to 111, and then each draws £11,100 a month.

The division almost writes itself once you pair the houses from the ends inward:

$$1 + 36 = 2 + 35 = 3 + 34 = \cdots = 18 + 19 = 37.$$

There are eighteen such pairs, each summing to 37. Hand each son three whole pairs: that is six houses summing to $3 \times 37 = 111$, exactly the equal share. Any way of dealing the eighteen pairs three-to-a-son works.

The hundred-house version is the same idea. The numbers 1 to 100 add to 5050, so each of the ten sons needs houses summing to 505. Pairing from the ends, $1 + 100 = 2 + 99 = \cdots = 50 + 51 = 101$, gives fifty pairs of 101; five pairs to a son is ten houses summing to $5 \times 101 = 505$. The trick works whenever the houses pair into equal sums and the pairs deal out evenly among the heirs.

Puzzle 20

A Will of Diamonds

✠

A man left 114 identical diamonds, with instructions that they go to his four sons: one third to the eldest, one quarter to the second, one fifth to the third and one sixth to the youngest. No diamond was to be cut. But a third of 114 is 38, a quarter is not a whole number at all, and the executor was stuck. Can the will be carried out?

Solution

The fractions in the will do not add up to one:

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}=\frac{20+15+12+10}{60}=\frac{57}{60}=\frac{19}{20}.$$

So the father never disposed of his whole estate, only nineteen twentieths of it, and that is what makes 114 awkward to divide. The cure is the old trick of the seventeen camels. Borrow six diamonds from a friend, making 120. Now the shares come out whole:

$$\frac{120}{3}=40,\quad \frac{120}{4}=30,\quad \frac{120}{5}=24,\quad \frac{120}{6}=20,$$

and these total $40+30+24+20=114$. Give the sons 40, 30, 24 and 20 diamonds, hand the six borrowed stones back, and every son has received rather more than the letter of the will would have given him, with not a single diamond cut. The borrowed six were never needed except to make the arithmetic whole, which is exactly why 120, the lowest common multiple of the denominators, is the number to reach for.

Puzzle 21

Four Weights for Forty

A stone weighing forty pounds is broken into four pieces. A shopkeeper finds that with these four pieces, placed in either pan of a balance, he can weigh out any whole number of pounds from 1 to 40. What are the weights of the four pieces?

Solution

The four pieces weigh 1, 3, 9 and 27 pounds, the powers of three, and indeed $1 + 3 + 9 + 27 = 40$.

The reason they suffice is that a balance lets each weight do one of three things: sit in the pan opposite the load, sit in the same pan as the load, or stay off the scale altogether. So the amounts you can weigh are exactly the totals

$$(\pm 1) + (\pm 3) + (\pm 9) + (\pm 27),$$

where each of the four pieces may be added (placed opposite the load), subtracted (placed beside it), or left out. This is what is called balanced ternary, counting in threes with the digits -1, 0 and 1. Every whole number from -40 to 40 can be written in this form in exactly one way, so in particular each amount from 1 to 40 can be weighed, and uniquely. For example $5 = 9 - 3 - 1$: put the 9 opposite the load, the 3 and the 1 beside it.

Puzzle 22

One Light Column

There are ten columns of pennies, ten coins to a column. Every coin is genuine and weighs 10 grams, except that one whole column is counterfeit, each of its coins weighing 9 grams. You have an accurate scale but may use it for only a single weighing.

How do you find the counterfeit column, and with how few coins?

Solution

Number the columns 0 to 9, and take that many coins from each: none from column 0, one from column 1, two from column 2, and so on up to nine from column 9. That is $0 + 1 + 2 + \cdots + 9 = 45$ coins on the scale at once.

Were every coin genuine, the pile would weigh 450 grams. Each counterfeit coin is a gram light, and the number of counterfeit coins in the pile is exactly the number of the offending column. So the shortfall below 450 grams, read straight off the scale, names the column: a deficit of 7 grams means column 7, and a deficit of nothing at all means column 0, from which we took no coin. One weighing settles it, using 45 coins. (Taking 1 through 10 coins instead also works, but costs ten more coins for no extra information.)

Puzzle 23

The Ages of Three Children

Two old friends meet. One asks the other the ages of his three children. "The product of their ages is 36," he says. "That doesn't tell me enough," says the first. "Their sum is the number on that house across the road." The first looks, thinks, and says "I still can't tell." "Ah," says the father, "the eldest is learning the piano." At once the first friend knows all three ages. What are they?

Solution

List the ways three whole-number ages can multiply to 36, with the sum of each:

$$\begin{array}{ll} (1,1,36):38 & (1,2,18):21 \\ (1,3,12):16 & (1,4,9):14 \\ (1,6,6):13 & (2,2,9):13 \\ (2,3,6):11 & (3,3,4):10 \end{array}$$

The first friend can see the house number, that is, the sum, yet still cannot decide. So the sum must fail to pick out a single line, which happens for one value only: the sum 13, shared by $(1,6,6)$ and $(2,2,9)$. Every other sum appears just once and would have given the answer away.

The deciding clue is that there is an *eldest* child. Of the two possibilities, $(1,6,6)$ has no single eldest, its two older children being the same age, while $(2,2,9)$ does. So the ages are 2, 2 and 9. The remark about the piano carries no information of its own; it matters only because the word "eldest" tells us one child stands alone at the top.

Puzzle 24

Numbers Made of Their Own Factorials

The number 145 has a curious property: add the factorials of its digits and you get the number back,

$$1! + 4! + 5! = 1 + 24 + 120 = 145.$$

Find every number, beyond the trivial 1 and 2, equal to the sum of the factorials of its digits.

Solution

There are just two more, and the complete list is

$$1, \quad 2, \quad 145, \quad 40585.$$

The last checks out as $4! + 0! + 5! + 8! + 5! = 24 + 1 + 120 + 40320 + 120 = 40585$ (recall $0! = 1$).

The reason the hunt ends is a race between two quantities. A number with d digits is at least 10^{d-1}. The most its digit factorials can ever total is $d \times 9! = 362880\,d$, since $9!$ is the biggest single-digit factorial. For $d = 8$ the number is already at least $10^7 = 10{,}000{,}000$, while its digit factorials can reach at most $8 \times 362880 = 2{,}903{,}040$, which is smaller. From eight digits on the number always outruns the sum, so no large example can exist, and a finite check settles the rest. Such numbers are called *factorions*, and in base ten these four are all there are.

Puzzle 25

Ten in a Row, None Prime

Primes thin out as numbers grow, leaving longer and longer stretches with none at all. Find ten consecutive whole numbers, not one of them prime.

Solution

The first such run begins at 114:

$$114, 115, 116, 117, 118, 119, 120, 121, 122, 123,$$

none of which is prime. In fact the gap runs a little further, all the way to 126, since 113 and 127 are the nearest primes on either side.

What is worth knowing is that runs of any length can be produced to order. For a chosen n, look at the $n - 1$ numbers

$$n! + 2, \quad n! + 3, \quad \dots, \quad n! + n.$$

Each is composite, because k divides $n!$ for every k from 2 to n, and so k divides $n! + k$ as well. Taking $n = 11$ gives ten consecutive composites $11! + 2, \dots, 11! + 11$, though they are enormous. So prime-free stretches as long as you please always exist; it is only the *smallest* ten-in-a-row that happens to start at the modest 114.

Puzzle 26

A Power of Every Kind

Find a number, somewhere between 3 and three thousand million million million (that is 3×10^{21}), that is at once a perfect square, a perfect cube, a perfect fourth power, a perfect fifth power and a perfect sixth power.

Solution

To be a square and a cube and a fourth, fifth and sixth power all together, a number must be a perfect Lth power where L is a common multiple of 2, 3, 4, 5 and 6. The least such L is their lowest common multiple,

$$\operatorname{lcm}(2, 3, 4, 5, 6) = 60,$$

so we want a sixtieth power. A sixtieth power is automatically all the rest, since 60 is a multiple of each: for instance it is the square of a thirtieth power and the cube of a twentieth.

The smallest sixtieth power above 3 is

$$2^{60} = 1{,}152{,}921{,}504{,}606{,}846{,}976 \approx 1.15 \times 10^{18},$$

which sits comfortably below the stated ceiling. The next, 3^{60}, is about 4.2×10^{28}, far beyond it. So 2^{60} is the one and only answer in range.

PUZZLE 27

His Age in a Square Year

A man was exactly X years old in the year X^2. How old was he in 1981?

Solution

We need a perfect square that could be a year within one lifetime of 1981. The squares near that era are

$$43^2 = 1849, \qquad 44^2 = 1936, \qquad 45^2 = 2025.$$

Of these, 1936 is the one that lets a person also be alive in 1981: a man who was 44 in 1936 was born in $1936 - 44 = 1892$, and in 1981 he was $1981 - 1892 = 89$ years old.

The trick is that perfect squares are sparse, only one every couple of decades in this range, so the single condition "X years old in year X^2" fixes the birth year almost completely. The same puzzle posed today turns on the next square: someone who is 45 in 2025 was born in 1980.

PUZZLE 28

One Before, One After

✠

Find a five-digit number with this property: writing a 1 after it gives a number three times as large as writing a 1 before it.

Solution

Call the number N. Writing a 1 after it appends a digit, making $10N + 1$; writing a 1 before it puts a 1 in the hundred-thousands place, making $100000 + N$. The condition is

$$10N + 1 = 3\,(100000 + N),$$

so $10N+1 = 300000+3N$, giving $7N = 299999$ and $N = 42857$. And indeed

$$428571 = 3 \times 142857.$$

That 142857 should surface here is no coincidence. It is the repeating block of $\frac{1}{7} = 0.\overline{142857}$, the best known of the cyclic numbers, and our answer 42857 is simply 142857 with its leading 1 struck off.

Puzzle 29

A Number That Counts Itself

✠

Fill ten boxes, labelled $0, 1, 2, \ldots, 9$, with single digits so that the digit in box k says how many times k appears in the ten-digit string you have written. So if box 3 holds a 2, the whole number must contain exactly two 3s.

Solution

There is exactly one such number:

6210001000.

Reading it against its own boxes: box 0 holds 6, and the number does contain six 0s; box 1 holds 2, and there are two 1s; box 2 holds 1, and there is a single 2; box 6 holds 1, and there is a single 6; every other box holds 0, and those digits indeed do not appear.

A quick way to corner it is to notice that the ten box-entries must add up to 10, because together they count each of the number's ten digits exactly once. Here $6 + 2 + 1 + 1 = 10$, as it must. That balance condition, with the requirement that the entries describe themselves, leaves 6210001000 as the only possibility.

Puzzle 30

The Coconuts and the Monkey

✠

Five castaways spend a day gathering coconuts into one pile, meaning to share them out next morning. A monkey watches. In the night one man wakes, distrusts the others, and divides the pile into five equal heaps; one coconut is left over, which he tosses to the monkey. He hides his own heap and pushes the other four back into a pile. One by one each of the other four does exactly the same: divides into five, finds one left over for the monkey, hides a fifth, restores the rest. In the morning the pile that remains divides evenly among the five, with none left over. What is the smallest number of coconuts they could have started with?

Solution

The smallest pile is 3121.

The clean way to see it uses a borrowed idea: a pile of -4 coconuts. Watch what the night-time operation does to such a pile. Take away one for the monkey, leaving -5; remove a fifth of that, namely -1, and four heaps of -1 remain, a pile of -4 again. So -4 is left utterly unchanged by the whole procedure. It follows that if the real pile N is such that $N + 4$ is divisible by enough fives, the arithmetic stays whole at every step. Five nights of dividing call for five factors of five, so set

$$N + 4 = 5^5 = 3125, \qquad N = 3121.$$

Then each man in turn finds a pile one more than a multiple of five, and after all five have taken their share the morning pile is $4^5 - 4 = 1020$, which divides evenly by five into 204 each. No smaller pile works.

The same trick handles the variations. For four castaways the smallest pile is 765; and for the harder case of seven castaways who leave three coconuts for the monkeys at each stage, it is 2470611.

Puzzle 31

The House in the Middle

✠

On a long street the houses are numbered $1, 2, 3, \ldots$ in order. A man living there noticed that the sum of all the house numbers below his own was exactly equal to the sum of all the numbers above it. His house number was even and lay between 100 and 999. What was it, and how many houses were on the street?

Solution

Let his house be number n and the last house on the street be N. The condition is

$$1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + N.$$

Add $1+2+\cdots+n$ to both sides and the right becomes the whole sum $1 + \cdots + N$, while the left becomes twice $1 + \cdots + (n - 1)$ plus n. Tidying up, the condition is exactly

$$n^2 = \frac{N(N + 1)}{2},$$

so his house number squared must be a triangular number. A number that is at once a perfect square and triangular is rare; the pairs (n, N) run

$$(1, 1),\ (6, 8),\ (35, 49),\ (204, 288),\ (1189, 1681),\ \ldots$$

The only even n between 100 and 999 is 204, paired with $N = 288$. So he lived at house 204 on a street of 288 houses. As a check, $1+\cdots+203 = 20706$, and $205+\cdots+288 = 41616-20910 = 20706$ as well.

Puzzle 32

Numbers That Reappear in Their Squares

✠

A three-digit number has this odd habit: the last three digits of its square are the number itself, in the same order. Find it. How many three-digit numbers behave this way?

Solution

There are two: 376 and 625, since

$$376^2 = 141\,376, \qquad 625^2 = 390\,625.$$

To see why these and no others, we want a three-digit n with n^2 ending in n, that is $n^2 - n = n(n-1)$ divisible by $1000 = 8 \times 125$. Now n and $n-1$ are consecutive, so they share no common factor; whatever powers of 2 and 5 make up the 1000 cannot be split between them, so the whole 8 goes into one of the two and the whole 125 into the other. That gives two interesting cases:

- 8 divides n while 125 divides $n-1$. So n is a multiple of 8 that is one more than a multiple of 125, and the only such number below 1000 is $n = 376$ (indeed $376 = 8 \times 47$ and $375 = 125 \times 3$).
- 125 divides n while 8 divides $n-1$. So n is a multiple of 125 that is one more than a multiple of 8, which gives $n = 625$ (since $625 = 125 \times 5$ and $624 = 8 \times 78$).

The only other possibilities, where 1000 divides n or $n-1$ outright, give numbers ending in 000 or 001, not a genuine three-digit answer. So 376 and 625 are the only ones. Such numbers are called *automorphic*. Pleasingly $376 + 625 = 1001$, and the habit persists into more digits: $625^2 = 390625$ ends in 0625, and $9376^2 = 87909376$.

Puzzle 33

A Cube and a Square

Find two whole numbers such that the difference of their squares is a perfect cube, while the difference of their cubes is a perfect square.

Solution

The numbers 6 and 10 do both at once:

$$10^2 - 6^2 = 100 - 36 = 64 = 4^3,$$
$$10^3 - 6^3 = 1000 - 216 = 784 = 28^2.$$

The difference of the squares is the cube of 4, and the difference of the cubes is the square of 28.

Puzzle 34

Threading Thirty-One and Seventy-Three

Find whole numbers m and n, allowing them to be negative, so that

$$31m + 73n = 1.$$

Solution

Run Euclid's algorithm on the two numbers, then unwind it. Dividing repeatedly,

$$73 = 2\cdot 31 + 11, \quad 31 = 2\cdot 11 + 9, \quad 11 = 1\cdot 9 + 2, \quad 9 = 4\cdot 2 + 1,$$

which confirms the two numbers share no common factor, so a solution exists. Now read the chain backwards, expressing the final remainder 1 in terms of earlier ones:

$$1 = 9 - 4\cdot 2 = 5\cdot 9 - 4\cdot 11 = 5\cdot 31 - 14\cdot 11 = 33\cdot 31 - 14\cdot 73.$$

So $m = 33$, $n = -14$, and indeed $31\cdot 33 - 73\cdot 14 = 1023 - 1022 = 1$. Every solution has the form $m = 33 + 73t$, $n = -14 - 31t$ for a whole number t, since one can shift a multiple of $73 \cdot 31$ between the two terms.

Puzzle 35

Eight Queens

Place eight queens on a chessboard so that no two attack each other: no two in the same row, the same column, or the same diagonal.

Solution

Take the rows in turn, from the bottom, and put the queen in columns

$$1, 5, 8, 6, 3, 7, 2, 4.$$

No column is repeated, so no two queens share a file. For the diagonals, note that two queens lie on a common diagonal exactly when their row-plus-column totals agree, or their row-minus-column differences agree; running through the eight placements, all eight sums are different and all eight differences are different, so no diagonal holds two queens. Drawing the board with row 8 at the top:

·	·	·	*Q*	·	·	·	·
·	*Q*	·	·	·	·	·	·
·	·	·	·	·	·	*Q*	·
·	·	*Q*	·	·	·	·	·
·	·	·	·	·	*Q*	·	·
·	·	·	·	·	·	·	*Q*
·	·	·	·	*Q*	·	·	·
Q	·	·	·	·	·	·	·

There are 92 ways in all to place the eight, which reduce to just 12 essentially different arrangements once rotations and mirror images of the board are counted as one.

Puzzle 36

Weighed Two at a Time

Five girls weigh themselves in pairs, trying every one of the ten possible couples. The pair weights, in pounds, come out as

$$114,\ 116,\ 118,\ 120,\ 121,\ 122,\ 123,\ 124,\ 125,\ 129.$$

No two girls weigh the same. What is the difference between the heaviest and the lightest?

Solution

Call the five weights $a < b < c < d < e$. Each girl is weighed against the other four, so each weight appears in four of the ten pair-totals. The ten totals therefore add up to $4(a+b+c+d+e)$, and since they sum to 1212,

$$a + b + c + d + e = \frac{1212}{4} = 303.$$

The lightest pairing is the two lightest girls, $a + b = 114$, and the heaviest is the two heaviest, $d + e = 129$. Subtracting these from the total leaves the middle girl,

$$c = 303 - 114 - 129 = 60.$$

The second-smallest total is $a + c = 116$, so $a = 56$ and then $b = 58$; the second-largest is $c+e = 125$, so $e = 65$ and then $d = 64$. The five weigh $56, 58, 60, 64, 65$ pounds, and the heaviest exceeds the lightest by $65 - 56 = 9$ pounds.

Puzzle 37

Squares of All the Digits

1. How many perfect squares use each of the digits 1 to 9 exactly once?
2. How many use all ten digits 0 to 9 exactly once?
3. Write 123456789 as the difference of two squares.

Solution

Parts 1 and 2. There are 30 squares using the digits 1 to 9 once each, the smallest being $11826^2 = 139854276$ and the largest $30384^2 = 923187456$. Allowing the zero as well, there are 87 squares using all ten digits once each, the smallest being $32043^2 = 1026753849$. (One checks these by running through the squares in the relevant range; both counts are settled.)

Part 3. Any odd number is a difference of two consecutive squares, since $(k+1)^2 - k^2 = 2k + 1$. As 123456789 is odd, set $2k + 1 = 123456789$, so $k = 61728394$ and

$$123456789 = 61728395^2 - 61728394^2.$$

Puzzle 38

Every Digit, Every Divisor

✠

1. Find the smallest number that uses each of the ten digits 0 to 9 once and is divisible by every one of $1, 2, 3, \ldots, 9$.
2. Find the smallest such ten-digit number divisible by every whole number from 2 to 18.

Solution

To be divisible by all of 1 to 9 is to be divisible by their lowest common multiple, which is 2520. So we hunt for the smallest ten-digit number, using each digit once, that is a multiple of 2520. It is

$$1234759680 = 2520 \times 489984.$$

For the second part the lowest common multiple of 2 through 18 is 12252240, and the smallest pandigital multiple of it is

$$2438195760 = 12252240 \times 199.$$

A small bonus: every number using all ten digits once is automatically divisible by 9, because its digits add to $0+1+\cdots+9 = 45$, itself a multiple of 9.

Puzzle 39

Adding a Million Numbers at Once

✠

What is the sum of all the different nine-digit numbers that can be made by using each of the digits 1 to 9 exactly once?

Solution

There are $9! = 362880$ such numbers, and no one would add them one by one. By symmetry each digit spends equal time in each of the nine places: it sits in the units place in $8! = 40320$ of the arrangements, in the tens place in another 40320, and so on. So each column, units through hundred-millions, has the same total,

$$(1 + 2 + \cdots + 9) \times 40320 = 45 \times 40320 = 1814400.$$

The whole sum is that column-total written into every place at once, that is multiplied by 111111111:

$$1814400 \times 111111111 = 201{,}599{,}999{,}798{,}400.$$

Puzzle 40

Matched at Every Power

Find two sets of six whole numbers, each below 100, sharing no member, such that the two sets have the same sum, the same sum of squares, the same sum of cubes, the same sum of fourth powers and the same sum of fifth powers.

Solution

The two sets

$$\{0, 5, 6, 16, 17, 22\} \qquad \text{and} \qquad \{1, 2, 10, 12, 20, 21\}$$

agree at every power from the first to the fifth:

$$\begin{aligned} &\text{sum} = 66, && \text{squares} = 1090, \\ &\text{cubes} = 19998, && \text{fourth powers} = 385234, \\ &\text{fifth powers} = 7632966, \end{aligned}$$

and they have no number in common. Such a matched pair is called a Prouhet-Tarry-Escott pair, and this is the smallest one that holds all the way up to fifth powers.

Puzzle 41

ADE plus ODO

In the addition below, each letter stands for a digit, and different letters stand for different digits. Find them.

$$\text{ADE} + \text{ODO} = \text{DEED}.$$

Solution

Two three-digit numbers add to the four-digit DEED, and the largest two three-digit numbers can make is under 2000, so the leading digit is 1: thus $D = 1$.

Now read the columns from the right. In the tens column the digits are D and D, so $D + D + c_1 = 2 + c_1$, where c_1 is whatever carried out of the units. This has to end in the tens digit of the answer, which is E, with no carry onward (since $2+c_1$ is at most 3). So $E = 2+c_1$. The units column reads $E+O$, ending in $D = 1$; that forces a carry, $c_1 = 1$, and $E + O = 11$. Then $E = 2 + 1 = 3$, so $O = 8$. Finally the hundreds column $A + O$ must give E with a carry of 1 into the thousands (which supplies the leading $D = 1$), so $A + O = E + 10 = 13$, giving $A = 5$. The unique solution is

$$513 + 818 = 1331.$$

Puzzle 42

What the Post Office Cannot Make

A shop sells stamps in only two values, 15 pence and 22 pence. Using any number of each, many totals can be made exactly, but not all: you cannot make 1p, or 7p, or 30p. What is the largest amount that cannot be made exactly?

Solution

The answer is 293 pence. Beyond it, every amount can be made; 293 itself cannot, and it is the last that defeats you.

The reason such a largest value exists at all is that 15 and 22 share no common factor. Once you can make some run of 15 consecutive amounts, you can make everything above it, simply by adding more 15p stamps to climb in steps of fifteen. For two coprime values m and n the largest unreachable total works out to

$$mn - m - n = 15 \times 22 - 15 - 22 = 330 - 37 = 293,$$

a tidy formula worth knowing.

That 293 really cannot be made is worth proving outright. Suppose $293 = 15a + 22b$ with a, b whole numbers, none negative. Add $15 + 22$ to both sides: the right becomes $15(a+1) + 22(b+1)$, and the left becomes $293 + 37 = 330 = 15 \times 22$. So

$$15(a+1) + 22(b+1) = 15 \times 22.$$

Now $15(a+1)$ is a multiple of 15, and the right side 15×22 is too, so the remaining term $22(b+1)$ must also be a multiple of 15. Since 22 and 15 share no common factor, it is $b+1$ that has to supply the factor of 15, so $b+1$ is at least 15. Swapping the roles of 15 and 22, the same reasoning forces $a+1$ to be at least 22. But then the left side is at least $15 \times 22 + 22 \times 15 = 2 \times 330$, far more than the 330 it is supposed to equal. The contradiction shows no such a, b exist, so 293 is unreachable.

Everything past it, on the other hand, is reachable. The next fifteen totals can each be made, for instance $294 = 15 \times 2 + 22 \times 12$ and $295 = 15 \times 5 + 22 \times 10$, and once fifteen consecutive totals are in hand, adding one more 15p stamp to each climbs through everything beyond in steps of fifteen.

Puzzle 43

The Greatest with Three Digits

Using only three digits, and any mathematical signs you like, what is the greatest number you can write? Roughly how many digits would it have?

Solution

The largest is the power tower

$$9^{9^9},$$

read from the top down: first 9^9, then 9 raised to that. It dwarfs the alternatives. Writing the digits side by side gives only 999; multiplying or taking ordinary powers like $(9^9)^9 = 9^{81}$ stays far smaller; the height of a tower beats them all. Note that 9^{9^9} means $9^{(9^9)}$, not $(9^9)^9$, and the difference is enormous: the exponent is $9^9 = 387{,}420{,}489$.

So the number is 9 raised to nearly 387 million. Its number of digits is about

$$9^9 \times \log_{10} 9 \approx 387{,}420{,}489 \times 0.954 \approx 370 \text{ million.}$$

Printed at a few digits to the centimetre it would run for hundreds of kilometres.

Puzzle 44

The Only Magic Hexagon

Nineteen circles are arranged in a hexagon, with rows of three, four, five, four and three. Place the numbers 1 to 19 in them so that every straight line of circles, in all three directions, adds to the same total.

Solution

First, the total is forced. The five horizontal rows, of three, four, five, four and three circles, between them cover all nineteen circles exactly once, and the numbers 1 to 19 add up to $\frac{19\times20}{2} = 190$. Five rows sharing that sum equally gives a common total of $190/5 = 38$. (This already shows the figure 38, not some smaller number, is the only one that can work.) And it can indeed be achieved:

```
        3       17      18
    19      7       1       11
16      2       5       6       9
    12      4       8       14
        10      13      15
```

Every row reads 38: the top row $3 + 17 + 18$, the long middle row $16 + 2 + 5 + 6 + 9$, and likewise each of the slanting lines in the other two directions.

The remarkable part is the answer to "find a second arrangement." There is none. Apart from turning this one by rotating or reflecting the board, no other magic hexagon exists, of this or any other size, using the natural numbers in order. The single example above is the only one there is, which makes it one of the rarest objects in recreational arithmetic.

Puzzle 45

Three Urns

✠

The first urn holds 1 white ball and 2 black; the second holds 2 white and 1 black; the third holds 2 white and 2 black. A blindfolded man moves one ball from the first urn to the second, then one ball from the second to the third, then draws a ball from the third. Is it an even chance that the ball he draws is white?

Solution

It is not quite even. The chance of white is $\frac{31}{60}$, a shade above one half.

Follow the two transfers. The ball leaving the first urn is white with probability $\frac{1}{3}$, black with probability $\frac{2}{3}$. That changes the second urn before the next draw:

- With probability $\frac{1}{3}$ the second urn becomes 3 white, 1 black, so the ball passed on to the third is white with probability $\frac{3}{4}$.
- With probability $\frac{2}{3}$ it becomes 2 white, 2 black, so the ball passed on is white with probability $\frac{1}{2}$.

So the ball arriving at the third urn is white with probability

$$\frac{1}{3} \cdot \frac{3}{4} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.$$

The third urn, originally 2 white and 2 black, gains this ball and then has five balls. Its white count is 2 plus the newcomer, so the chance the final draw is white is

$$\frac{1}{5}\left(2 + \frac{7}{12}\right) = \frac{1}{5} \cdot \frac{31}{12} = \frac{31}{60} \approx 0.517.$$

The blindfold tilts the odds very slightly towards white.

Puzzle 46

One Hundred from Nine Digits

✠

Using each of the digits 1 to 9 exactly once, and only plus signs and a fraction bar, write an expression equal to 100.

Solution

A neat one is

$$91 + \frac{5742}{638} = 91 + 9 = 100,$$

in which the digits $9, 1, 5, 7, 4, 2, 6, 3, 8$ are precisely 1 to 9, each used once. There are several others in the same spirit, for instance

$$81+\frac{7524}{396} = 81+19 = 100 \qquad \text{and} \qquad 82+\frac{3546}{197} = 82+18 = 100,$$

each again using every digit once. The knack is to pick the whole-number part, see how much is missing from 100, and then arrange the leftover digits into a fraction equal to that shortfall.

Puzzle 47

A Thousand Pounds in Bags

✠

A man brings £1000, all in single pound notes, to a bank and asks for it to be split among bags so that, whatever whole sum he later asks for, the teller can hand him some of the bags, unopened, totalling exactly that amount. What is the smallest number of bags that will do, and how many of them hold an odd number of pounds?

Solution

Ten bags suffice, and no fewer. Fill them with

$$1,\ 2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256 \quad \text{and} \quad 489,$$

the powers of two up to 256, with the last bag taking whatever remains, $1000 - 511 = 489$.

The powers of two are the key. Any whole amount from 0 to 511 is the sum of a unique selection of $1, 2, 4, \dots, 256$, which is just that amount written in binary. So every request up to 511 can be met from the first nine bags. For anything from 489 to 1000, hand over the 489 bag and make up the rest, between 0 and 511, from the others. Between them the two ranges, 0 to 511 and 489 to 1000, cover every pound up to 1000 with no gap.

Why not fewer than ten? With k bags there are only 2^k possible selections, and they must produce at least the 1000 distinct totals $1, 2, \dots, 1000$. Since $2^9 = 512 < 1000 \le 1024 = 2^{10}$, nine bags cannot manage and ten can. As for odd bags, of the ten amounts only 1 and 489 are odd, so exactly two bags hold an odd number of pounds.

Puzzle 48

An Ancient Magic Square

Place the numbers 1 to 16 in a four-by-four grid so that every row, every column and both main diagonals add to the same total. The old Indian square below does this and a good deal more besides. What is the total, and what are the "and more" properties?

Solution

The numbers 1 to 16 add up to $\frac{16\times17}{2} = 136$. Four rows share that sum equally, so each must total $136/4 = 34$. The square

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

has every row, every column and both diagonals summing to 34. But more is true, and all of it can be checked by eye:

- the four corner cells, $7 + 14 + 9 + 4 = 34$;
- the central two-by-two block, $13 + 8 + 3 + 10 = 34$, and likewise each two-by-two block at the four corners;
- the "broken" diagonals that wrap around the edges, such as $12 + 8 + 5 + 9 = 34$ and $1 + 11 + 16 + 6 = 34$.

A square whose broken diagonals also share the magic total is called *pandiagonal*, and this one is the most celebrated example, carved on a temple wall in Khajuraho. Many four-by-four magic squares exist, but only the pandiagonal ones carry all these extra balances at once.

Puzzle 49

The Counting Competition

A teacher writes a whole number and its square on the board and asks three pupils to count the total number of digits in the two numbers together. Anil says 999, Brijesh says 1000, Chandra says 1001. The teacher says that one of them cannot possibly be right, whatever the number was. Which one, and why?

Solution

Brijesh, with 1000, must be wrong.

Suppose the number has d digits. Then its square has either $2d$ or $2d-1$ digits: squaring roughly doubles the length, dropping one digit only when the leading figures are small enough not to carry into a new place. So the combined count is either

$$d + 2d = 3d \qquad \text{or} \qquad d + (2d-1) = 3d-1.$$

Every possible answer is therefore a multiple of three, or one less than a multiple of three. Now $999 = 3\times333$ is a multiple of three, and $1001 = 3\times334-1$ is one less than a multiple of three, so both could occur. But 1000 is neither: it is one *more* than the multiple 999, a count of the forbidden kind. So Brijesh's total can never arise, and he gets no prize.

Puzzle 50

Bananas and Pears

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A man bought nine bananas and six pears. He could not recall the prices, but remembered that each banana cost one penny more than each pear, and that the whole sum he paid for the pears was the figure he paid for the bananas with its two digits reversed. He paid less than a pound, that is under a hundred pence, for each kind. How much did he pay altogether?

Solution

Let a pear cost p pence, so a banana costs $p+1$. The bananas came to $9(p+1)$ pence and the pears to $6p$ pence, and we are told the pear figure is the banana figure reversed, with both under 100.

Testing the few pear prices that keep both totals to two digits, $p = 6$ works and nothing else does: the bananas cost $9 \times 7 = 63$ pence and the pears $6 \times 6 = 36$ pence, and 36 is indeed 63 written backwards. So altogether he paid

$$63 + 36 = 99 \text{ pence,}$$

a penny short of a pound.

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