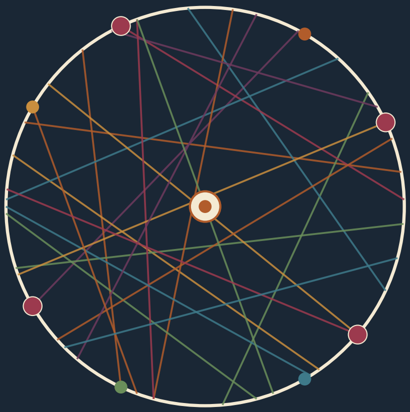
CURATED MATHEMATICS

# *Geometric Probability*



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LONDON

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This volume is a curated gallery of classical results in geometric probability, the corner of mathematics that asks "what is the chance that…?" about shapes drawn at random. The topics themselves are not new; most have been studied since Buffon, Bertrand, or Sylvester. The contribution of this book is a fresh exposition of each one, with full proofs at the level of a first-year undergraduate and figures redrawn at print quality.

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# *A Note on This Volume*

This book is a gallery of geometric probability. Each chapter takes one classical question, the chance that three random sticks form a triangle, that two random chords cross, that four random points in a triangle are in convex position, and gives a full, pictorial proof.

The field has a long and rich history. Georges-Louis Leclerc, Comte de Buffon, posed his *problem of the needle* in 1733: a needle of length $\ell$ dropped on a floor ruled with parallel lines at spacing $d \geq \ell$ crosses a line with probability $2\ell/(\pi d)$. Joseph Bertrand introduced his *paradox of the random chord* in 1889 to make the point that “random” needs a measure. James Joseph Sylvester posed his *four-point problem* in 1864. J. G. Wendel gave the sign-flip argument that determines the probability $n$ random points on a sphere contain the centre in 1962.

The topics here are not new, and most appear in the standard reference collections of the field, including Maurice Kendall and P. A. P. Moran’s *Geometrical Probability* (Griffin, 1963), Luis Santaló’s *Integral Geometry and Geometric Probability* (Cambridge University Press, 1976), and Daniel A. Klain and Gian-Carlo Rota’s *Introduction to Geometric Probability* (Cambridge University Press, 1997). The contribution of this volume is a fresh exposition of each topic, complete proofs at undergraduate level, and original figures.

The chapters can be read in any order. Each defines its central object from scratch and proves its central result on its own terms. The five parts group the gems thematically, area ratios, symmetry, integration, number-theoretic bridges, measuretheoretic subtleties, and three-dimensional analogues, but no chapter depends on another.

Every chapter ends with a short Python snippet that verifies the analytic answer by Monte Carlo simulation. Across all 29 chapters the simulated values match the closed forms to within sampling error; we found no discrepancies. The code relies only on `numpy` and can be pasted into an ordinary Python 3 interpreter. Readers are encouraged to modify the snippets, change parameters, vary the sampling rule, try new questions, and so turn the book into a small interactive laboratory.

# *Contents*

# Part I

# Warmups: area ratios in parameter space

## CHAPTER 1

# *Three sticks forming a triangle*

**PROBLEM 1.1.** Let $a, b, c$ be independent uniform random variables in $[0, 1]$. What is the probability that sticks of lengths $a, b, c$ form a triangle?

*Solution.* Three positive lengths form a triangle iff each is less than the sum of the other two. The parameter space is the unit cube $[0, 1]^3$, with total volume 1.

The triangle inequality $a < b + c$ *fails* in the region $\{a \geq b + c\}$, which is the corner tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (1, 1, 0), (1, 0, 1)$. Its volume is $\frac{1}{6}$ (one-sixth of the unit cube). By symmetry, the analogous tetrahedra for $b \geq a + c$ and $c \geq a + b$ also have volume $\frac{1}{6}$ each. These three bad regions are disjoint: at most one side can exceed the sum of the other two.

Hence the bad volume is $3 \cdot \frac{1}{6} = \frac{1}{2}$, and the good volume — the triangle region — is $1 - \frac{1}{2} = \frac{1}{2}$. The probability of forming a triangle is

$$\boxed{\mathbb{P} = \frac{1}{2}.}$$

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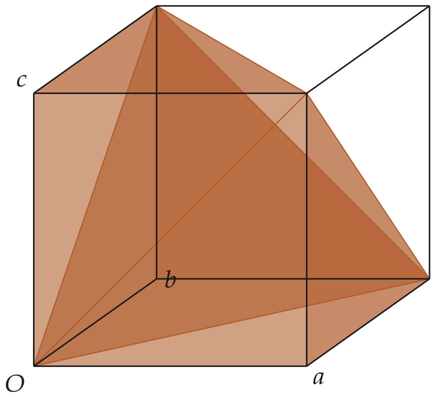

Figure 1.1: The unit cube $[0,1]^3$ of stick lengths $(a, b, c)$. The three shaded corner tetrahedra (copper) are the configurations where one stick is at least as long as the other two combined; each has volume $\frac{1}{6}$. The remaining volume $\frac{1}{2}$ is the triangle region.

*Monte Carlo verification.*

```
import numpy as np
a, b, c = np.random.rand(3, 10**7)
p = ((a+b>c) & (b+c>a) & (c+a>b)).mean()
print(f"sim: {p:.5f}   exact: 0.5")
# sim: 0.50005   exact: 0.5
```

## Chapter 2

# *The longest of three pieces*

**Problem 2.1.** A unit stick is snapped at two independent uniform random points. What is the probability that the longest of the three resulting pieces has length greater than $\frac{1}{2}$?

*Solution.* Let the break points be at positions $x, y \in [0,1]$, independent uniform. The three pieces have lengths $\min(x,y)$, $|x-y|$, and $1-\max(x,y)$. The longest piece exceeds $\frac{1}{2}$ iff *some* piece exceeds $\frac{1}{2}$:

$$\begin{aligned}\mathbb{P}(\text{longest} > \tfrac{1}{2}) = \mathbb{P}\big(&\min(x,y) > \tfrac{1}{2} \text{ or} \\ &|x-y| > \tfrac{1}{2} \text{ or} \\ &1-\max(x,y) > \tfrac{1}{2}\big).\end{aligned}$$

The three events are *mutually exclusive*: if $\min(x,y) > \frac{1}{2}$ then both $x, y > \frac{1}{2}$, forcing $|x-y| < \frac{1}{2}$ and $\max(x,y) > \frac{1}{2}$; similarly if $\max(x,y) < \frac{1}{2}$ then $|x-y| < \frac{1}{2}$. So we may add the three probabilities directly.

Each event corresponds to a region of the unit square:

- $\{\min(x,y) > \frac{1}{2}\} = [\frac{1}{2}, 1]^2$, area $\frac{1}{4}$,
- $\{\max(x,y) < \frac{1}{2}\} = [0, \frac{1}{2}]^2$, area $\frac{1}{4}$,
- $\{|x-y| > \frac{1}{2}\}$ is the union of two corner triangles (upper-left and lower-right), each of area $\frac{1}{8}$, total area $\frac{1}{4}$.

Summing,

$$\boxed{\mathbb{P}(\text{longest} > \tfrac{1}{2}) = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{4}.}$$

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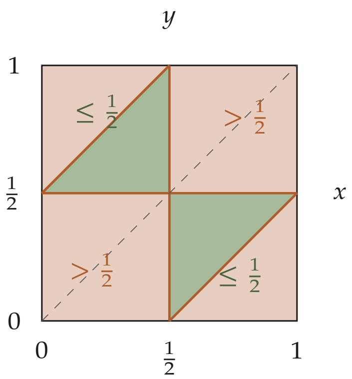

Figure 2.1: The unit square of break points $(x, y)$. The central sage region — where every piece has length $\leq \frac{1}{2}$ — is a bow-tie of area $\frac{1}{4}$. Its complement (copper), of area $\frac{3}{4}$, is the region where the longest piece exceeds $\frac{1}{2}$.

*Monte Carlo verification.*

```
import numpy as np
x, y = np.random.rand(2, 10**7)
pieces = np.stack([np.minimum(x,y), np.abs(x-y),
     0.694120.360780.16864↪ 1-np.maximum(x,y)])
p = (pieces.max(axis=0) > 0.5).mean()
print(f"sim: {p:.5f}   exact: 0.75")
# sim: 0.74991   exact: 0.75
```

CHAPTER 3

# *Closer to centre than to any side*

**PROBLEM 3.1.** A point $P$ is chosen uniformly at random in the unit square $[-\frac{1}{2}, \frac{1}{2}]^2$. What is the probability that $P$ is closer to the centre $O = (0, 0)$ than to every side?

*Solution.* Split the square into four triangles by its two diagonals. Consider the top triangle $T = \{(x, y) : y \geq |x|\}$: here the closest side is the top side $y = \frac{1}{2}$, at distance $\frac{1}{2} - y$.

In $T$, the point $(x, y)$ is closer to $O = (0, 0)$ than to the top side iff

$$\sqrt{x^2 + y^2} \leq \tfrac{1}{2} - y.$$

Both sides are non-negative on $T$, so squaring is reversible: the condition becomes $x^2 + y^2 \leq \frac{1}{4} - y + y^2$, i.e., $y \leq \frac{1}{4} - x^2$. This is the region under a downward parabola with focus $O$ and directrix $y = \frac{1}{2}$.

The parabola $y = \frac{1}{4} - x^2$ meets the boundary line $y = x$ of $T$ where $x = \frac{1}{4} - x^2$, giving $x^* = \frac{\sqrt{2}-1}{2}$. On $T$, the region *both* inside $T$ (so $y \geq |x|$) and below the parabola is

$$R_T = \{(x, y) : |x| \leq x^*,\ |x| \leq y \leq \tfrac{1}{4} - x^2\}.$$

Its area equals

$$\text{area}(R_T) = 2\int_0^{x^*} \left(\tfrac{1}{4} - x^2 - x\right) dx = 2\left[\tfrac{x^*}{4} - \tfrac{(x^*)^3}{3} - \tfrac{(x^*)^2}{2}\right].$$

With $x^* = (\sqrt{2}-1)/2$ we have $(x^*)^2 = (3-2\sqrt{2})/4$ and $(x^*)^3 = (5\sqrt{2} - 7)/8$. Substituting and simplifying,

$$\text{area}(R_T) = \frac{4\sqrt{2} - 5}{12}.$$

By the fourfold symmetry of the square, the full region where $P$ is closer to $O$ than to every side has area

$$4 \cdot \frac{4\sqrt{2}-5}{12} = \frac{4\sqrt{2}-5}{3}.$$

Since the unit square has area 1,

$$\boxed{\mathbb{P} = \frac{4\sqrt{2}-5}{3} \approx 0.219.}$$

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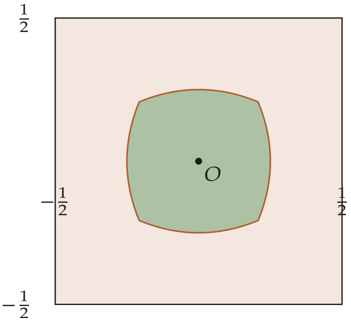

Figure 3.1: The region (sage) inside the unit square where a point is closer to the centre than to any side. The boundary consists of four parabolic arcs (copper), each with focus at the centre and one side as directrix. The region's area is $\frac{4\sqrt{2}-5}{3}$.

*Monte Carlo verification.*

```
import numpy as np
x, y = np.random.rand(2, 10**7) - 0.5
d_centre = np.sqrt(x*x + y*y)
d_side = 0.5 - np.maximum(np.abs(x), np.abs(y))
p = (d_centre < d_side).mean()
print(f"sim: {p:.5f}   exact: {(4*np.sqrt(2)-5)/3:.5f}")
# sim: 0.21916   exact: 0.21895
```

## CHAPTER 4

# A random quadratic with real roots

**PROBLEM 4.1.** Let $p, q$ be independent uniform random variables in $[-1, 1]$. What is the probability that the quadratic $x^2 + 2px + q = 0$ has real roots?

*Solution.* The quadratic has real roots iff its discriminant is non-negative, i.e., $(2p)^2 - 4q \geq 0$, or equivalently $q \leq p^2$. The parameter space is the square $[-1, 1]^2$, of area 4.

The region $\{(p, q) : q \leq p^2\}$ inside the square is everything *below* the parabola $q = p^2$. Its area is

$$\int_{-1}^{1} (p^2 - (-1))\, dp = \int_{-1}^{1} (p^2 + 1)\, dp = \tfrac{2}{3} + 2 = \tfrac{8}{3}.$$

Hence the probability is

$$\boxed{\mathbb{P} = \frac{8/3}{4} = \tfrac{2}{3}.}$$

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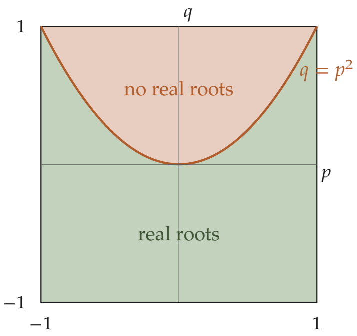

Figure 4.1: Random quadratic $x^2 + 2px + q$ with $(p, q)$ uniform in $[-1, 1]^2$. The quadratic has real roots iff $(p, q)$ lies below the parabola $q = p^2$ (sage region, area $\frac{8}{3}$). The complementary region (copper) has area $\frac{4}{3}$. The ratio gives $\mathbb{P} = \frac{2}{3}$.

*Monte Carlo verification.*

```
import numpy as np
p, q = np.random.uniform(-1, 1, (2, 10**7))
real_roots = p**2 >= q
print(f"sim: {real_roots.mean():.5f}   exact: {2/3:.5f}")
# sim: 0.66659   exact: 0.66667
```

## CHAPTER 5

# *The meeting problem*

**PROBLEM 5.1.** Alice and Bob each arrive at a café at an independent uniform random time between noon and 1:00 PM. Each waits 15 minutes before leaving. What is the probability that they meet?

*Solution.* Let $x, y \in [0, 1]$ be the arrival times of Alice and Bob (in units of an hour, with $0 =$ noon). The parameter space is the unit square $[0, 1]^2$ with total area 1. Alice and Bob meet iff their arrival times are within a quarter hour of each other:

$$|x - y| \leq \tfrac{1}{4}.$$

The region $\{|x-y| > \frac{1}{4}\}$ is the union of two corner triangles: one above the line $y = x + \frac{1}{4}$, and one below the line $y = x - \frac{1}{4}$. Each is a right triangle with legs of length $\frac{3}{4}$, so each has area $\frac{1}{2} \cdot (\frac{3}{4})^2 = \frac{9}{32}$. Their combined area is $\frac{9}{16}$.

Hence the meeting region has area $1 - \frac{9}{16} = \frac{7}{16}$, and

$$\boxed{\mathbb{P}(\text{meet}) = \frac{7}{16}.}$$

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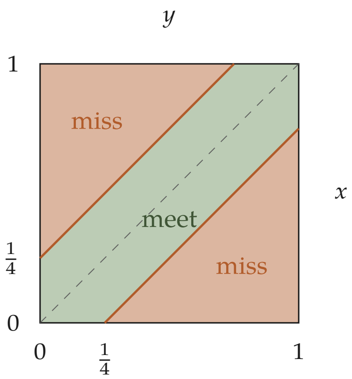

Figure 5.1: The meeting problem. The unit square of arrival times $(x, y)$; Alice and Bob meet iff $|x - y| \leq \frac{1}{4}$ (sage band). The two corner triangles (copper) with total area $\frac{9}{16}$ are the "miss" region. Meeting probability $= \frac{7}{16}$.

The meeting problem generalises: if Alice waits $\alpha$ hours and Bob waits $\beta$ hours (both in a 1-hour window), the meeting region becomes an L-shape and the probability is $1-(1-\alpha)(1-\beta)$ if $\alpha+\beta \leq 1$. For $\alpha = \beta = \frac{1}{4}$, this gives $1-(3/4)^2 = 7/16$.

*Monte Carlo verification.*

```
import numpy as np
x, y = np.random.rand(2, 10**7)
p = (np.abs(x - y) <= 0.25).mean()
print(f"sim: {p:.5f}   exact: {7/16:.5f}")
# sim: 0.43746   exact: 0.43750
```

CHAPTER 6

# *Buffon's coin (franc-carreau)*

✠

**PROBLEM 6.1.** A floor is tiled with squares of side $d$. A coin of radius $r < d/2$ is tossed at random onto the floor. What is the probability that the coin lands entirely within one tile (i.e., does not overlap any tile boundary)?

*Solution.* By translation symmetry, the position of the coin is determined by the location of its centre relative to the nearest tile corner; we may take this centre to lie uniformly in a single $d \times d$ tile.

The coin lies entirely within the tile iff its centre lies in the inner square $[r, d-r]^2$. This inner square has side $d - 2r$ and area $(d-2r)^2$. The parameter tile has area $d^2$, so

$$\boxed{\mathbb{P}(\text{entirely within one tile}) = \left(\frac{d-2r}{d}\right)^2 = \left(1 - \frac{2r}{d}\right)^2.}$$

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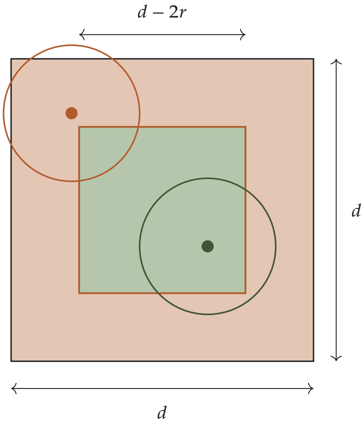

Figure 6.1: Buffon's coin problem (*franc-carreau*, 1733). The coin lands within one tile iff its centre falls in the smaller sage square of side $d-2r$; the outer copper border (of width $r$) is the "boundary-touching" region.

This is historically the *first* geometric-probability problem ever posed — Buffon (1733) called it *le jeu du franc-carreau*, a children's game played by tossing a coin at floor tiles and winning if the coin lay within one tile. Buffon's needle (Chapter **??**) appeared as a more celebrated sequel some years later.

*Monte Carlo verification.*

```
import numpy as np
r, d, N = 0.2, 1.0, 10**7
u, v = np.random.rand(2, N) * d  # coin centre in tile
in_tile = (u >= r) & (u <= d-r) & (v >= r) & (v <= d-r)
print(f"sim: {in_tile.mean():.5f}   exact: {(1 -
    0.694120.360780.16864↪ 2*r/d)**2:.5f}")
# sim: 0.36003   exact: 0.36000
```

CHAPTER 7

# Broken stick, second break on the longer piece

✠

**PROBLEM 7.1.** A unit stick is broken at a uniform random point $X \in [0,1]$. Of the two resulting pieces, the longer one is then broken at a uniform random point along its length. What is the probability that the three resulting pieces form a triangle?

*Solution.* By symmetry between the two outcomes $X \le \frac{1}{2}$ and $X \ge \frac{1}{2}$, we may condition on $X \le \frac{1}{2}$; the other case is handled identically by reversing the stick. So take $X \le \frac{1}{2}$, giving a shorter piece of length $X$ and a longer piece of length $1 - X$. Now break the longer piece: let $Y \in [0, 1 - X]$ be uniform, giving two pieces of lengths $Y$ and $1 - X - Y$.

The three lengths are $X$, $Y$, and $1 - X - Y$. They form a triangle iff each is less than the sum of the other two, equivalently iff each is less than $\frac{1}{2}$:

$$X < \tfrac{1}{2}, \quad Y < \tfrac{1}{2}, \quad 1 - X - Y < \tfrac{1}{2}.$$

The first is automatic under our conditioning. The remaining two are:

$$Y < \tfrac{1}{2}, \qquad Y > \tfrac{1}{2} - X.$$

Given $X$, $Y$ is uniform on $[0, 1 - X]$, so

$$\mathbb{P}(\text{triangle} \mid X) = \frac{\frac{1}{2} - \left(\frac{1}{2} - X\right)}{1 - X} = \frac{X}{1 - X}.$$