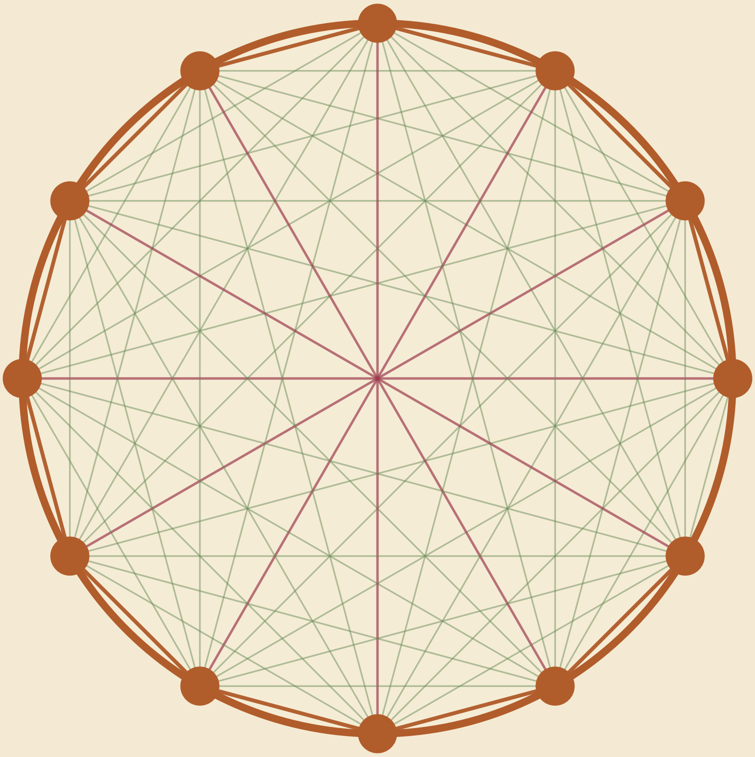
A COURSE IN CLASSICAL ALGEBRA



How to See an Equation

The art of algebraic reasoning

Vamshi Jandhyala

LONDON

A COURSE IN CLASSICAL ALGEBRA

How to See an Equation

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Classical algebra through problems

Vamshi Jandhyala

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The historical attributions in this book are, to the best of the author's knowledge, accurate, and all mathematical claims are proved in full.

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For every young reader who has ever sat
in front of a quadratic and felt
the floor tilt.

You are not alone. The puzzle is older
than you, and so is the answer.

Mathematics knows no races or geographic boundaries;
for mathematics, the cultural world is one country.

David Hilbert, 1930

That which is sought may yet be restored.

al-Khwārizmī, Baghdad, c. 820

A theorem is a proof a person has been willing to finish.

on the first page of a working notebook

A Note to the Reader

This book is for the reader who wants to *do* algebra, not merely read it. It is written for the ambitious and motivated student who has met the usual school syllabus and wants to meet what lies beyond.

Its purpose is twofold. First, to teach rigorous mathematical reasoning: how to read a problem carefully, how to choose a technique, how to write a proof that someone else can follow. Second, to display the beautiful and powerful techniques that classical algebra has accumulated: the Euclidean algorithm, Viète's formulas, the factor theorem, Newton's identities, the classical inequalities, and the rest. Every problem in the book is a small piece of that inheritance.

Each chapter is a carefully chosen set of problems, grouped into short movements by technique. Almost every problem reveals a distinct idea, and the techniques rarely repeat; the few foundational results that recur, such as the arithmetic-geometric mean inequality, return as reusable tools rather than as repeated exercises. The problems are classical, picked from the problem-solving tradition that has stood up over decades of use, and they favour concrete questions ("find the value of", "solve for x", "compute the remainder when …") over abstract theorems.

Every problem is followed by a discussion. The discussion names the central technique, gives the shortest honest proof or computation, works the steps through in full, and marks the trap that catches the unwary.

Part I, *Foundations*, assembles the instruments: proof, number systems, algebraic identities, functions, inequalities, induction, and exponentials and logarithms. Part II, *Algebra*, puts them to work: quadratic and polynomial equations, complex numbers, sequences, counting, the binomial theorem, the classical inequalities, and functional equations. Each chapter requires only those that precede it; each exists to unlock the next.

The chapters are best read in order. A reader who needs a specific tool, however, should feel free to jump: the dependency graph is explicit, and every definition is restated where it first bears weight.

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Part I

Foundations

CHAPTER 1

Proof and logic

DIRECT PROOF AND CONTRAPOSITIVE

PROBLEM 1.1. Let n be any integer. Show that $n^5 - n$ is divisible by 30.

Technique. Direct proof: work forward from the hypothesis using algebra and a little case-checking.

Discussion. Since $30 = 2 \cdot 3 \cdot 5$, and $2, 3, 5$ have no common factor among themselves, it is enough to show that $n^5 - n$ is divisible by each of $2, 3$, and 5 separately. (If an integer is divisible by three numbers that share no common factor, it must be divisible by their product.)

Factor:

$$n^5 - n \;=\; n(n^2-1)(n^2+1) \;=\; (n-1)\,n\,(n+1)(n^2+1).$$

Divisible by 2 *and by* 3*.* Among any two consecutive integers, one is even; among any three, one is a multiple of 3. So the first three factors $(n-1) \cdot n \cdot (n+1)$ already contribute both a factor of 2 and a factor of 3.

Divisible by 5*.* We check the five possible remainders when n is divided by 5.

- If n leaves remainder 0: then n itself is divisible by 5.
- If n leaves remainder 1: then $n - 1$ is divisible by 5.
- If n leaves remainder 4: then $n + 1$ is divisible by 5.

- If n leaves remainder 2: then n^2 leaves remainder 4, so n^2+1 leaves remainder 5, which is 0. So n^2+1 is divisible by 5.
- If n leaves remainder 3: similarly, n^2 leaves remainder 9, which is 4, so $n^2 + 1$ is again divisible by 5.

In every case, at least one of the four factors of n^5-n is divisible by 5.

Combining: $n^5 - n$ is divisible by $2, 3$, and 5, and therefore by 30. □

What to take away. A "divisible by 30" target breaks up into "divisible by 2", "divisible by 3", and "divisible by 5". Each piece is attacked separately, and then the pieces are multiplied back together. The clever step here is the factoring $n^5 - n = (n-1) \cdot n \cdot (n+1) \cdot (n^2+1)$; without it, the problem is much harder.

Problem 1.2. Let $n \geq 2$ be an integer. Show that if $2^n - 1$ is prime, then n itself must be prime.

Technique. Contrapositive: the statement "if P, then Q" says the same thing as "if not Q, then not P." Sometimes the second form is easier to prove.

Discussion. The original statement is: if $2^n - 1$ is prime, then n is prime. The contrapositive is: if n is *not* prime, then $2^n - 1$ is *not* prime. We prove the contrapositive.

Suppose n is not prime. Then $n = ab$ for some integers a, b with $1 < a, b < n$. We use a useful algebraic identity (essentially the sum of a geometric progression): for any number x,

$$x^b - 1 = (x-1)(x^{b-1} + x^{b-2} + \cdots + x + 1).$$

Set $x = 2^a$. Then $x^b = 2^{ab} = 2^n$, and the identity becomes

$$2^n - 1 = (2^a - 1) \cdot (2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1).$$

We have expressed $2^n - 1$ as a product of two integers. Let us check that both are greater than 1:

- $2^a - 1 \geq 2^2 - 1 = 3$, since $a \geq 2$.

- The second factor is a sum of b positive terms, each at least 1, so it is at least $b \geq 2$.

So $2^n - 1$ is the product of two integers, each bigger than 1, and so $2^n - 1$ is not prime.

This proves the contrapositive, and therefore the original statement. □

What to take away. The contrapositive trades the original question for a slightly different one. Here the original asks us to say something about when $2^n - 1$ is prime, and primes don't hand over their structure. The contrapositive asks us to say something about when n is composite, which is easy, since we can just write $n = ab$. That shift is the whole trick.

A warning: the converse "if n is prime, then $2^n - 1$ is prime" is false. For instance $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23 \times 89$ is not. So the property "n prime" is necessary for $2^n - 1$ to be prime, but it is not enough.

PROOF BY CONTRADICTION

PROBLEM 1.3. Show that $\log_2 3$ is irrational.

Technique. Proof by contradiction: assume the thing you want to disprove, and derive something impossible.

Discussion. Suppose, for a contradiction, that $\log_2 3$ is rational. Then it can be written as p/q for some positive integers p and q (both positive because $\log_2 3$ is a positive number, being greater than 1). By the definition of logarithm,

$$2^{p/q} = 3, \qquad \text{and so} \qquad 2^p = 3^q.$$

Now look at the two sides. The left side, 2^p, is a power of 2, so it is even (since $p \geq 1$). The right side, 3^q, is a power of 3, so it is odd. An integer cannot be even and odd at the same time. Contradiction.

Therefore $\log_2 3$ cannot be rational. □

What to take away. Proof by contradiction is the only tool that lets you assume something whose truth you do not yet know. You assume it, chase the consequences, and hope to reach nonsense. Here the nonsense was "an integer is both even and odd". Once the nonsense appears, the assumption must have been wrong.

Problem 1.4. Show that there are infinitely many prime numbers.

Technique. Contradiction together with an explicit construction: assume the list is finite, then build an integer the list cannot explain.

Discussion. This is Euclid's argument, from around 300 BC.

Suppose, for a contradiction, that there are only finitely many primes, and list them all: $p_1, p_2, \ldots, p_k$. Form the new integer

$$N = p_1 p_2 \cdots p_k + 1.$$

$N > 1$, so N has some prime factor; call it p. Is p on the list? If $p = p_i$ for some i, then p_i divides both N and $p_1 p_2 \cdots p_k$, so p_i must also divide their difference, $N - p_1 p_2 \cdots p_k = 1$. But no prime divides 1. So p is *not* on the list.

But we assumed the list contained every prime. Contradiction. Therefore the list of primes cannot be finite. □

What to take away. Notice the shape of the argument: assume the thing exists (a finite list), then *build* an object (the number N) that cannot fit. The construction $N = p_1 p_2 \cdots p_k + 1$ is the creative step; without it, we would just be staring at an assumption. Many impossibility proofs follow the same template.

Proof by cases

Problem 1.5. Show that no integer of the form $4k + 3$ (where k is an integer) can be written as a sum of two perfect squares.

Technique. Proof by cases on remainders: check each possible remainder on division by a well-chosen number.

Discussion. We want to show: for every pair of integers a, b, the sum $a^2 + b^2$ is never of the form $4k + 3$. The key is to look at remainders when we divide by 4.

Step 1: what are the possible remainders of a square when divided by 4?

- If a is even, write $a = 2m$. Then $a^2 = 4m^2$, which leaves remainder 0.
- If a is odd, write $a = 2m + 1$. Then $a^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1$, which leaves remainder 1.

So every square, when divided by 4, leaves remainder 0 or remainder 1.

Step 2: what are the possible remainders of $a^2 + b^2$? The remainder of $a^2 + b^2$ on division by 4 is the sum of the individual remainders of a^2 and b^2 (possibly minus a multiple of 4 to keep it between 0 and 3). Since each of a^2, b^2 contributes 0 or 1, their sum contributes $0, 1,$ or 2. The remainder 3 is never produced.

So no number that leaves remainder 3 when divided by 4, in other words no number of the form $4k + 3$, can equal $a^2 + b^2$. □

What to take away. When a claim is about *every* integer, it is often useful to sort integers by their remainder on division by some carefully chosen number. The job of the proof is then reduced to checking finitely many cases. The creative step is picking the right divisor, in this case 4. A different choice (say 3 or 5) would not have worked.

Problem 1.6. Show that among any five integers, one can always pick three whose sum is divisible by 3.

Technique. Proof by cases, combined with the *pigeonhole principle*: if you put 5 items into 3 boxes, at least one box must contain at least $\lceil 5/3 \rceil = 2$ items.

Discussion. Sort the five given integers by their remainder on division by 3. The possible remainders are $0, 1, 2$: three boxes. Five integers, three boxes; how can they be distributed?

Case 1: some box contains three or more of the integers. Pick any three of them. They all leave the same remainder r when divided by 3. So their sum leaves remainder $3r$, which is a multiple of 3. Done.

Case 2: no box contains three of the integers. Then every box contains at most 2 integers. But five items split into three boxes of size at most 2 can only be distributed as $2 + 2 + 1$. In particular, every box contains at least one integer. So we can pick one integer from the "remainder 0" box, one from the "remainder 1" box, and one from the "remainder 2" box. Their sum leaves remainder $0 + 1 + 2 = 3$, which is $0 \pmod 3$, a multiple of 3. Done.

In both cases, three of the five integers sum to a multiple of 3. □

What to take away. The proof had two different ways of finding the three integers, depending on how the inputs were distributed. Splitting into cases here is not a mere convenience: the three integers we picked are genuinely different in the two cases. The pigeonhole principle ("some box must be full") was the tool that forced one of the two cases to hold.

Existence and counterexamples

Problem 1.7. Show that there exist irrational numbers a and b such that a^b is rational.

Technique. Existence without a specific example: we will prove that a and b exist without being able to say exactly what they are.

Discussion. Consider the number $(\sqrt{2})^{\sqrt{2}}$. There are only two possibilities: either it is rational, or it is irrational. We need not decide which, because the proof works either way.

Case 1: $(\sqrt{2})^{\sqrt{2}}$ is rational. Set $a = \sqrt{2}$ and $b = \sqrt{2}$. Both a and b are irrational (the classical proof, standard textbook). And $a^b = (\sqrt{2})^{\sqrt{2}}$ is rational by the assumption of this case. Done.

Case 2: $(\sqrt{2})^{\sqrt{2}}$ is irrational. Set $a = (\sqrt{2})^{\sqrt{2}}$ and $b = \sqrt{2}$. Then a is irrational (by the assumption of this case), and b is irrational. Compute:

$$a^b = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2})^2 = 2,$$

which is rational. Done.

In both cases, there exist irrational a, b with a^b rational. □

What to take away. Notice the slightly unsettling fact: the argument itself never reveals which of the two cases is the true one. It does not, on its own, tell us whether $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational. (The value is in fact irrational, by the Gelfond–Schneider theorem, but that lies well beyond this book, and the existence proof has no need of it.) Nevertheless, we have proved that the pair (a, b) exists, even without the proof naming it. An existence proof does not have to produce the object; showing it must be there is sometimes enough.

Problem 1.8. Disprove the claim: "$n^2 - n + 41$ is a prime number for every positive integer n."

Technique. Disproof by counterexample: to refute a claim of the form "for every n, something holds," exhibit one n for which it fails.

Discussion. A universally quantified claim ("$\forall n, P(n)$") is disproved by a single n for which $P(n)$ fails. No general argument is required.

Try $n = 41$:

$$41^2 - 41 + 41 \;=\; 41^2 \;=\; 1681 \;=\; 41 \times 41.$$

This is not prime. So $n = 41$ is a counterexample, and the claim is false. □

What to take away. One concrete failure is a complete refutation. It is tempting to say things like "the formula must eventually fail" or "no formula can be prime for every input"; those are weaker than the truth, which is that the formula fails at a specific, namable n.

(The formula $n^2 - n + 41$ actually *is* prime for every n from 1 to 40, a string of 40 correct cases. This is why it is a famous trap. A large number of examples supporting a claim is not a proof of the claim, no matter how many examples.)

UNIQUENESS AND "IF AND ONLY IF"

PROBLEM 1.9. Let a, b be integers with $a > 0$. Show that there is at most one way to write

$$b \;=\; q\,a + r, \qquad 0 \le r < a,$$

with q and r integers.

Technique. Proof of uniqueness: assume there are two such ways, and show they must actually be the same.

Discussion. Suppose there are two representations of b in the required form:

$$b \;=\; q_1\,a + r_1 \;=\; q_2\,a + r_2, \qquad \text{with } 0 \le r_1, r_2 < a.$$

We want to show $q_1 = q_2$ and $r_1 = r_2$.

Rearrange: $(q_1 - q_2)a = r_2 - r_1$. The left side is a multiple of a; therefore so is the right.

Now use the bound on the remainders. Both r_1 and r_2 lie in the range $0 \le r < a$, so their difference satisfies

$$-a \;<\; r_2 - r_1 \;<\; a.$$

In that range $(-a, a)$, the *only* multiple of a is 0. So $r_2 - r_1 = 0$, meaning $r_1 = r_2$. Going back to the equation $(q_1 - q_2)a = 0$, and dividing by a (which is allowed since $a > 0$), we get $q_1 = q_2$ too.

So the two representations are the same, and the representation is unique. □

What to take away. The standard shape of a uniqueness proof: assume two answers, subtract them, show their difference must be zero. Here the key ingredient was the bound on r. Because r was forced into a narrow range, the difference $r_2 - r_1$ was too small to be a non-zero multiple of a. Narrow bounds plus divisibility conditions are a powerful combination.

Problem 1.10. Show that a positive integer n can be written as a difference of two squares $n = a^2 - b^2$ (with a, b non-negative integers) if and only if n is *not* of the form $4k + 2$.

Technique. "If and only if" means two separate claims, and we prove each in turn.

Discussion. The claim is a two-way statement, so we prove it in two parts.

Part 1 (⇒): if n is a difference of two squares, then n is not of the form $4k + 2$.

Suppose $n = a^2 - b^2$. Factor: $n = (a - b)(a + b)$. Now notice that $a - b$ and $a + b$ have the same parity: their sum is $2a$ (even), so either both are even or both are odd. Two sub-cases:

- Both odd. Then n is a product of two odd numbers, so n is odd.
- Both even. Then write $a - b = 2u$ and $a + b = 2v$; so $n = 4uv$, a multiple of 4.

Either way, n is odd or a multiple of 4, and in particular n is not of the form $4k + 2$.

Part 2 (⇐): if n is not of the form $4k + 2$, then n is a difference of two squares.

We give explicit a and b in two cases.

- n is odd. Let $a = \frac{n+1}{2}$ and $b = \frac{n-1}{2}$. (Both are integers, since n is odd.) Check: $a^2 - b^2 = (a-b)(a+b) = 1 \cdot n = n$.
- n is a multiple of 4, say $n = 4k$. Let $a = k+1$ and $b = k-1$. Check: $a^2 - b^2 = (a-b)(a+b) = 2 \cdot 2k = 4k = n$.

So in both cases we have exhibited a and b with $n = a^2 - b^2$.

Both parts proved, the "iff" is established. □

What to take away. An "if and only if" is really two separate statements, and usually the two halves need their own proofs. Here one half was a parity argument; the other was an explicit formula. It is not enough to prove only one direction, because the other could easily be false. When you can, *construct* the object (as we did in Part 2) rather than just assert it exists.

CHAPTER 2

Number systems

GREATEST COMMON DIVISORS AND LINEAR EQUATIONS IN INTEGERS

PROBLEM 2.1. Find $\gcd(2^{100} - 1, 2^{60} - 1)$.

Technique. Run the Euclidean algorithm *on the exponents*, using the identity $2^m - 1 \equiv 2^r - 1 \pmod{2^n - 1}$ where r is the remainder when m is divided by n.

Discussion. If $m = qn + r$ with $0 \le r < n$, then

$$2^m \;=\; (2^n)^q \cdot 2^r \;\equiv\; 1^q \cdot 2^r \;=\; 2^r \pmod{2^n - 1},$$

so $2^m - 1 \equiv 2^r - 1 \pmod{2^n - 1}$. Therefore $\gcd(2^m - 1, 2^n - 1) = \gcd(2^n - 1, 2^r - 1)$. The exponents follow exactly the same shrinking process as the Euclidean algorithm applied to (m, n).

Applying to $(100, 60)$:

$$100 = 1 \cdot 60 + 40, \qquad 60 = 1 \cdot 40 + 20, \qquad 40 = 2 \cdot 20 + 0.$$

So $\gcd(100, 60) = 20$, and therefore

$$\gcd(2^{100} - 1, 2^{60} - 1) \;=\; 2^{20} - 1 \;=\; 1\,048\,575.$$

□

What to take away. The identity $\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1$ is the key. The same identity holds with 2 replaced by any integer $a \geq 2$. It is why every Mersenne prime $2^p - 1$ forces p itself to be prime: if p factored as $p = mn$ non-trivially, the identity would give $2^m - 1$ as a proper divisor of $2^p - 1$.

Problem 2.2. Find all pairs (x, y) of positive integers satisfying $7x + 11y = 100$.

Technique. A linear equation $ax + by = c$ in integers has solutions iff $\gcd(a, b) \mid c$. Find one solution using the Euclidean algorithm in reverse; all others differ from it in a standard way.

Discussion. Since $\gcd(7, 11) = 1$ divides 100, solutions exist. Run the Euclidean algorithm on $(11, 7)$ in reverse to express 1 as $7x + 11y$:

$$11 = 1 \cdot 7 + 4, \qquad 7 = 1 \cdot 4 + 3, \qquad 4 = 1 \cdot 3 + 1.$$

Back-substitute:

$$1 = 4 - 1 \cdot 3 = 4 - (7 - 4) = 2 \cdot 4 - 7 = 2(11 - 7) - 7 = 2 \cdot 11 - 3 \cdot 7.$$

So $(-3) \cdot 7 + 2 \cdot 11 = 1$. Multiply by 100:

$$(-300) \cdot 7 + 200 \cdot 11 = 100.$$

One integer solution is $(x_0, y_0) = (-300, 200)$.

The general integer solution adds $(11t, -7t)$ for any integer t (this is the "null-space" of the equation, since $11 \cdot 7 - 7 \cdot 11 = 0$):

$$x = -300 + 11t, \qquad y = 200 - 7t.$$

For both $x > 0$ and $y > 0$: $x > 0$ needs $t > 300/11 \approx 27.27$, so $t \geq 28$. $y > 0$ needs $t < 200/7 \approx 28.57$, so $t \leq 28$. So $t = 28$ is the only value, giving

$$(x, y) = (8, 4).$$

Check: $7 \cdot 8 + 11 \cdot 4 = 56 + 44 = 100$. □

What to take away. The general solution to a linear Diophantine equation is "one particular solution plus integer multiples of a shift". Once a single solution is found (via back-substitution through the Euclidean algorithm), restricting to positive x and y becomes a pair of inequalities on t.

Modular arithmetic

Problem 2.3. Find the last two digits of 7^{2024}.

Technique. The last two digits of an integer are its residue modulo 100. Find a small power of 7 that is $\equiv 1 \pmod{100}$, then reduce the exponent.

Discussion. Build up the residues of successive powers of 7 modulo 100:

$$7^1 \equiv 7, \qquad 7^2 \equiv 49, \qquad 7^3 \equiv 43, \qquad 7^4 \equiv 1.$$

So $7^4 \equiv 1 \pmod{100}$: the powers cycle with period 4.

Reduce the exponent: $2024 = 4 \cdot 506$, so

$$7^{2024} \;=\; (7^4)^{506} \;\equiv\; 1^{506} \;=\; 1 \pmod{100}.$$

The last two digits are 01. □

What to take away. For "last k digits of a^N" questions: find the order of a modulo 10^k (the smallest positive d with $a^d \equiv 1$), then reduce N modulo d. Here $d = 4$ and $2024 \equiv 0 \pmod 4$, so the answer is exactly $7^0 = 1$.

Problem 2.4. Find the largest positive integer k such that 5^k divides 100!.

Technique. Count the multiples of 5, 25, 125, … up to 100. Each multiple of 5^j contributes one extra factor of 5.

Discussion. $100! = 1 \cdot 2 \cdot 3 \cdots 100$. A particular integer $n \leq 100$ contributes j factors of 5 to this product, where 5^j is the exact power of 5 in n.

Count by "layer":

- Multiples of 5 in $\{1, \ldots, 100\}$: there are $\lfloor 100/5 \rfloor = 20$ of them (namely $5, 10, 15, \ldots, 100$). Each contributes at least one factor of 5.
- Multiples of 25: $\lfloor 100/25 \rfloor = 4$ of them $(25, 50, 75, 100)$. Each contributes an *additional* factor of 5 on top of the one already counted.

- Multiples of 125: $\lfloor 100/125 \rfloor = 0$.

Total factors of 5:

$$k \;=\; \lfloor 100/5 \rfloor + \lfloor 100/25 \rfloor + \cdots \;=\; 20 + 4 + 0 + \cdots \;=\; 24.$$

$$k \;=\; 24.$$

□

What to take away. This is Legendre's formula: the exact power of a prime p dividing $n!$ is $\sum_{j \geq 1} \lfloor n/p^j \rfloor$. It comes up whenever you need to count trailing zeros (the number of trailing zeros of 100! is $\min(v_2(100!), v_5(100!)) \;=\; 24$, since there are plenty of factors of 2 around), or when the divisibility of $\binom{n}{k}$ by a prime is in question.

DECIMALS AND RATIONALS

PROBLEM 2.5. Write the repeating decimal $0.\overline{123}$ (i.e., $0.123\,123\,123\,\ldots$) as a fraction in lowest terms.

Technique. Multiply by a power of 10 matching the period of repetition, subtract, and solve for the fraction.

Discussion. Let $x = 0.\overline{123}$. The block "123" has length 3, so multiply by $10^3 = 1000$:

$$1000x \;=\; 123.\overline{123}, \qquad x \;=\; 0.\overline{123}.$$

Subtract:

$$999x \;=\; 123, \qquad \text{so} \qquad x \;=\; \frac{123}{999}.$$

Reduce: $\gcd(123, 999)$. Note $123 = 3 \cdot 41$ and $999 = 3^3 \cdot 37$, so $\gcd = 3$. Dividing:

$$x \;=\; \frac{41}{333}.$$

□

What to take away. Every repeating decimal with period d can be written as (integer) / $(10^d - 1)$. Once in this form, reducing to lowest terms is a gcd computation. The converse statement, that every rational has an eventually repeating decimal, follows from the pigeonhole principle on remainders during long division: there are only finitely many possible non-zero remainders, so one must recur, starting a cycle.

Problem 2.6. Which of the following fractions have terminating decimal expansions?

$$\frac{1}{6}, \quad \frac{1}{8}, \quad \frac{1}{12}, \quad \frac{1}{15}, \quad \frac{1}{20}, \quad \frac{1}{25}, \quad \frac{1}{28}, \quad \frac{1}{35}, \quad \frac{1}{40}, \quad \frac{1}{50}.$$

Technique. A fraction p/q in lowest terms terminates iff the only primes dividing q are 2 and 5.

Discussion. Factor each denominator into primes:

$$6 = 2 \cdot 3, \quad 8 = 2^3, \quad 12 = 2^2 \cdot 3, \quad 15 = 3 \cdot 5, \quad 20 = 2^2 \cdot 5,$$

$$25 = 5^2, \quad 28 = 2^2 \cdot 7, \quad 35 = 5 \cdot 7, \quad 40 = 2^3 \cdot 5, \quad 50 = 2 \cdot 5^2.$$

Those with only 2s and 5s in their factorisation are 8, 20, 25, 40, 50. The rest contain a 3 or a 7.

Terminating:

$$\frac{1}{8} = 0.125, \quad \frac{1}{20} = 0.05, \quad \frac{1}{25} = 0.04,$$

$$\frac{1}{40} = 0.025, \quad \frac{1}{50} = 0.02.$$

The others are repeating: $1/6 = 0.1\overline{6}$, multiples of $1/7$ have period 6 (so $1/28$ and $1/35$ eventually settle into 6-cycles), and multiples of $1/3$ produce the $\overline{3}$ or $\overline{6}$ tails.

Why the rule holds. If $p/q = k/10^n$ for some integer k and $n \geq 0$, then q divides $10^n = 2^n \cdot 5^n$, so q's only possible prime factors are 2 and 5. Conversely, if $q = 2^a \cdot 5^b$, set $m = \max(a, b)$; multiply top and bottom by $2^{m-a} \cdot 5^{m-b}$ to get denominator 10^m, i.e., a terminating decimal. □

What to take away. The "terminating decimal" property is a statement about the denominator's relationship to the base. In base 10, it is 2s and 5s. In base $6 = 2 \cdot 3$, it would be 2s and 3s, and $1/5$ would no longer terminate.

SURDS AND IRRATIONALITY

PROBLEM 2.7. Find all integers n with $1 \leq n \leq 50$ for which $\sqrt{n}$ is rational.

Technique. $\sqrt{n}$ is rational if and only if n is a perfect square. Applied concretely: list the perfect squares in the given range.

Discussion. We first prove the criterion. Suppose $\sqrt{n} = p/q$ is rational, with $\gcd(p, q) = 1$. Then $nq^2 = p^2$. If $q > 1$, some prime r divides q; then $r \mid p^2$, and since r is prime, $r \mid p$, contradicting $\gcd(p, q) = 1$. So $q = 1$, meaning $\sqrt{n}$ is an integer, and n is its square.

So $\sqrt{n}$ rational $\iff$ n is a perfect square.

In the range $1 \leq n \leq 50$, the perfect squares are $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$, which is

$$n \in \{1, 4, 9, 16, 25, 36, 49\}.$$

(Note $8^2 = 64 > 50$.) □

What to take away. The statement "$\sqrt{n}$ is irrational unless n is a perfect square" is a clean generalisation of "$\sqrt{2}$ is irrational". The proof rests on the prime factorisation structure of integers: if n is not a perfect square, some prime appears to an odd power in its factorisation, and this odd-parity obstruction cannot be matched by a square p^2.

PROBLEM 2.8. Compute the exact value of

$$\sqrt{5 + 2\sqrt{6}} \; + \; \sqrt{5 - 2\sqrt{6}}.$$

Technique. Un-nest each radical by writing $5 \pm 2\sqrt{6}$ as a perfect square of the form $(\sqrt{a} \pm \sqrt{b})^2$.

Discussion. Look for positive reals a, b with $(\sqrt{a} + \sqrt{b})^2 = 5 + 2\sqrt{6}$. Expand: $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$. So we need $a + b = 5$ and $ab = 6$. These are Viète's conditions for the roots of $t^2 - 5t + 6 = (t - 2)(t - 3) = 0$; so $\{a, b\} = \{2, 3\}$.

Therefore

$$\sqrt{5 + 2\sqrt{6}} \; = \; \sqrt{2} + \sqrt{3}.$$

By the same logic (with − replacing +, which changes the middle sign on un-nesting):

$$\sqrt{5-2\sqrt{6}} \;=\; \sqrt{3}-\sqrt{2}$$

(not $\sqrt{2}-\sqrt{3}$, because the radical must be non-negative and $\sqrt{3}>\sqrt{2}$).

Add:

$$\sqrt{5+2\sqrt{6}} \;+\; \sqrt{5-2\sqrt{6}} \;=\; (\sqrt{2}+\sqrt{3})+(\sqrt{3}-\sqrt{2}) \;=\; 2\sqrt{3}.$$

□

What to take away. Nested radicals $\sqrt{c\pm 2\sqrt{d}}$ un-nest if you can find positive a, b with $a+b=c$ and $ab=d$; these are exactly Viète's conditions for $t^2-ct+d=0$. Once un-nested, the expression is a sum or difference of two simple surds, and algebra (as in this problem's addition step) tends to produce something compact.

ABSOLUTE VALUE

PROBLEM 2.9. Find all real numbers x satisfying

$$|x^2-4|+|x^2-9| \;=\; 5.$$

Technique. Recognise the left-hand side as $|u-4|+|u-9|$ for $u=x^2$, and use the identity $|u-a|+|u-b|\geq|a-b|$, with equality exactly when u lies between a and b.

Discussion. Let $u=x^2$. The triangle inequality gives

$$|u-4|+|9-u| \;\geq\; |(u-4)+(9-u)| \;=\; |5| \;=\; 5,$$

with equality iff $(u-4)$ and $(9-u)$ have the same sign. The second term $|9-u|$ equals $|u-9|$, so the left side of our equation is always ≥ 5, and equality holds iff u lies between 4 and 9.

So the equation $|u-4|+|u-9|=5$ is equivalent to $4\leq u\leq 9$. Translating to x: $4\leq x^2\leq 9$, i.e., $2\leq|x|\leq 3$.

$$x\in[-3,-2]\cup[2,3].$$

□

What to take away. This is a classic technique: when the equation is of the form $|u-a|+|u-b|=|a-b|$, the triangle inequality pins u to the interval $[a,b]$ in one step. There is no case analysis on the individual signs; a single inequality and its equality condition do all the work.

Problem 2.10. Solve $|x+1|+|x-1|+|x-2|=4$ for real x.

Technique. Split the real line at the points $-1,1,2$ where each absolute-value term changes sign. On each resulting interval the expression is linear.

Discussion. The break-points are $x=-1,1,2$, dividing the real line into four intervals.

Case $x \le -1$. Here $|x+1|=-(x+1)$, $|x-1|=-(x-1)=1-x$, $|x-2|=2-x$. Sum: $-(x+1)+(1-x)+(2-x)=2-3x$. Set equal to 4: $2-3x=4$, $x=-\frac{2}{3}$. But $-\frac{2}{3}>-1$, so this is not in the case's range. Reject.

Case $-1 \le x \le 1$. Here $|x+1|=x+1$, $|x-1|=1-x$, $|x-2|=2-x$. Sum: $(x+1)+(1-x)+(2-x)=4-x$. Set equal to 4: $x=0$. Since $0 \in [-1,1]$, this is a valid solution.

Case $1 \le x \le 2$. Here $|x+1|=x+1$, $|x-1|=x-1$, $|x-2|=2-x$. Sum: $(x+1)+(x-1)+(2-x)=x+2$. Set equal to 4: $x=2$. Since $2 \in [1,2]$, this is a valid solution.

Case $x \ge 2$. Here all three are +: $(x+1)+(x-1)+(x-2)=3x-2=4$, so $x=2$. Already found.

Combining:

$$x=0 \ \text{ or } \ x=2.$$

□

What to take away. The standard approach for equations or inequalities involving several absolute-value terms: identify the break-points, split the real line accordingly, solve the resulting linear equation on each interval, and check that each candidate lies in the correct interval. A solution is accepted only if it falls in the range where it was derived. This last step is the one most often missed.

CHAPTER 3

Algebraic expressions and identities

✠

FACTORING CLASSICAL FORMS

PROBLEM 3.1. Factor $x^4 + 4$ completely over the integers.

Technique. Sophie Germain's identity: add and subtract the same term to turn one part of the expression into a perfect square, then apply the difference of squares.

Discussion. At first sight $x^4 + 4$ looks like it has no factors. It has no real roots (since $x^4 \geq 0$ for all real x, so $x^4 + 4 \geq 4 > 0$), so it has no linear factors over $\mathbb{R}$, let alone over $\mathbb{Z}$. But a quartic can still factor as a product of two quadratics.

The trick: introduce an auxiliary $4x^2$ term, then take it back,

$$x^4 + 4 \;=\; x^4 + 4x^2 + 4 \;-\; 4x^2 \;=\; (x^2 + 2)^2 - (2x)^2.$$

Now apply the difference of squares $A^2 - B^2 = (A - B)(A + B)$ with $A = x^2 + 2$ and $B = 2x$:

$$(x^2 + 2 - 2x)(x^2 + 2 + 2x) \;=\; (x^2 - 2x + 2)(x^2 + 2x + 2).$$

Both quadratic factors have discriminant $4 - 8 = -4 < 0$, so they have no real roots and cannot be factored further over $\mathbb{Z}$.

$$x^4 + 4 \;=\; (x^2 - 2x + 2)(x^2 + 2x + 2).$$

☐

What to take away. When a quartic stubbornly refuses to factor, try adding and subtracting a term that turns one part of it into a perfect square. The general form, due to Sophie Germain, is

$$a^4 + 4b^4 = (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2).$$

With $b = 1$, this is the identity we just used.

Problem 3.2. Factor $a^5 + b^5$ completely as a polynomial in a and b.

Technique. For odd n, $a^n + b^n$ always has $(a + b)$ as a factor, because substituting $a = -b$ makes the expression vanish. Once $(a+b)$ is pulled out, the quotient has an alternating-sign pattern.

Discussion. Substitute $a = -b$ into $a^5 + b^5$:

$$(-b)^5 + b^5 = -b^5 + b^5 = 0.$$

So by the factor theorem (treating the expression as a polynomial in a with b as a parameter), $(a - (-b)) = (a + b)$ divides $a^5 + b^5$.

Divide and check: the quotient is a polynomial of degree 4 in a, b, and the pattern for odd n is known,

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}).$$

For $n = 5$,

$$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4).$$

To verify: expanding $(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$ produces $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$ (from the a terms) and $a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5$ (from the b terms). The cross terms all cancel, leaving $a^5 + b^5$. □

What to take away. For odd n, $a^n + b^n$ factors as $(a + b)$ times a polynomial with alternating signs. For even n, $a^n + b^n$ has no factor of the form $(a \pm b)$, and the quadratic $a^2 + b^2$ is irreducible over $\mathbb{R}$. That does not make the higher even powers irreducible, though: they can still split into real quadratics. For instance $a^4 + b^4 = (a^2 + \sqrt{2}\,ab + b^2)(a^2 - \sqrt{2}\,ab + b^2)$. The difference $a^n - b^n$, however, always has $(a - b)$ as a factor, with a positive-sign quotient, for every n.

SYMMETRIC POLYNOMIAL IDENTITIES

PROBLEM 3.3. Let a, b, c be non-zero real numbers with $a + b + c = 0$. Find the value of

$$\frac{a^3 + b^3 + c^3}{abc}.$$

Technique. The classical identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, applied with $a + b + c = 0$.

Discussion. There is a classical identity, verifiable by direct expansion,

$$a^3 + b^3 + c^3 - 3abc \ = \ (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

If $a+b+c = 0$, the right-hand side is zero, so $a^3+b^3+c^3-3abc = 0$, that is,

$$a^3 + b^3 + c^3 \ = \ 3abc.$$

Dividing by abc (which is non-zero by assumption):

$$\frac{a^3 + b^3 + c^3}{abc} \ = \ 3.$$

□

What to take away. The identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(\ldots)$ is one of the load-bearing identities of classical algebra. Whenever a problem carries the hypothesis $a+b+c = 0$ (or can be rearranged to read that way), expressions involving sums of cubes of a, b, c collapse dramatically.

PROBLEM 3.4. Three real numbers a, b, c satisfy

$$a + b + c = 3, \qquad a^2 + b^2 + c^2 = 5, \qquad a^3 + b^3 + c^3 = 7.$$

Find $a^4 + b^4 + c^4$.

Technique. Newton's identities link the *power sums* $p_k = a^k + b^k + c^k$ to the *elementary symmetric functions* $e_1 = a + b + c$, $e_2 = ab + bc + ca$, $e_3 = abc$. Once the elementary symmetric functions are known, each power sum is determined by a linear recurrence.

Discussion. Write $p_k = a^k + b^k + c^k$ and e_1, e_2, e_3 for the elementary symmetric functions. We are given $p_1 = 3, p_2 = 5$, $p_3 = 7$, and $e_1 = 3$.

Find e_2. From the identity $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$,

$$e_1^2 = p_2 + 2e_2, \qquad 9 = 5 + 2e_2, \qquad e_2 = 2.$$

Find e_3. Newton's identity for three variables at level $k = 3$ reads

$$p_3 = e_1 p_2 - e_2 p_1 + 3e_3.$$

Substitute:

$$7 = 3\cdot 5 - 2\cdot 3 + 3e_3 = 15 - 6 + 3e_3 = 9 + 3e_3, \qquad e_3 = -\tfrac{2}{3}.$$

Find p_4. Newton's identity at level $k = 4$ (and higher) has the form

$$p_k = e_1 p_{k-1} - e_2 p_{k-2} + e_3 p_{k-3}.$$

For $k = 4$:

$$p_4 = 3\cdot 7 - 2\cdot 5 + (-\tfrac{2}{3})\cdot 3 = 21 - 10 - 2 = 9.$$

□

What to take away. Newton's identities convert power sums to and from elementary symmetric functions. For three variables, once e_1, e_2, e_3 are pinned down, every p_k follows from the three-term recurrence $p_k = e_1 p_{k-1} - e_2 p_{k-2} + e_3 p_{k-3}$. This is the standard tool when a problem gives you a few power sums and asks for another.

Conjugates and rationalisation

Problem 3.5. Rationalise the denominator and simplify:

$$\frac{1}{1 - \sqrt[3]{2}}.$$

Technique. Use the identity $1-x^3=(1-x)(1+x+x^2)$ with $x=\sqrt[3]{2}$ to clear the cube root.

Discussion. For square roots, one rationalises by multiplying by the conjugate ($a-\sqrt{b}$ paired with $a+\sqrt{b}$ gives a^2-b). For cube roots, the situation is different: multiplying $(1-\sqrt[3]{2})$ by $(1+\sqrt[3]{2})$ gives $1-\sqrt[3]{4}$, which still contains a cube root. One needs a three-term companion.

Apply $1-x^3=(1-x)(1+x+x^2)$ with $x=\sqrt[3]{2}$:

$$(1-\sqrt[3]{2})(1+\sqrt[3]{2}+\sqrt[3]{4}) \;=\; 1-(\sqrt[3]{2})^3 \;=\; 1-2 \;=\; -1.$$

So multiplying numerator and denominator of the original expression by $1+\sqrt[3]{2}+\sqrt[3]{4}$:

$$\frac{1}{1-\sqrt[3]{2}} \;=\; \frac{1+\sqrt[3]{2}+\sqrt[3]{4}}{-1} \;=\; -(1+\sqrt[3]{2}+\sqrt[3]{4}).$$

☐

What to take away. To rationalise $1/(a-\sqrt[n]{b})$, multiply numerator and denominator by the companion factor that produces a^n-b. For $n=2$, the companion is a single conjugate; for $n=3$, a sum of three terms; for general n, a sum of n terms with appropriate signs and powers.

Problem 3.6. Compute the exact value of

$$(2+\sqrt{3})^5 \;+\; (2-\sqrt{3})^5.$$

Technique. Recognise $2+\sqrt{3}$ and $2-\sqrt{3}$ as conjugate roots of a quadratic with integer coefficients. Their power sums $p_n=\alpha^n+\beta^n$ satisfy a linear recurrence that produces an integer at every step.

Discussion. Set $\alpha=2+\sqrt{3}$ and $\beta=2-\sqrt{3}$. Then

$$\alpha+\beta \;=\; 4, \qquad \alpha\beta \;=\; 4-3 \;=\; 1.$$

So α,β are the roots of the quadratic $t^2-4t+1=0$.

Let $p_n=\alpha^n+\beta^n$. Multiplying the quadratic $\alpha^2-4\alpha+1=0$ by α^{n-2} gives $\alpha^n=4\alpha^{n-1}-\alpha^{n-2}$, and similarly for β. Adding yields the recurrence

$$p_n \;=\; 4p_{n-1}-p_{n-2}.$$

Starting values:

$$p_0 = 1 + 1 = 2, \qquad p_1 = \alpha + \beta = 4.$$

Now step up:

$$\begin{aligned} p_2 &= 4 \cdot 4 - 2 = 14, \\ p_3 &= 4 \cdot 14 - 4 = 52, \\ p_4 &= 4 \cdot 52 - 14 = 194, \\ p_5 &= 4 \cdot 194 - 52 = 724. \end{aligned}$$

□

What to take away. Any expression of the form $\alpha^n + \beta^n$, where α and β are the roots of a quadratic with rational (in particular integer) coefficients, is always rational, even though α and β individually may not be. The linear recurrence $p_n = (\alpha + \beta)p_{n-1} - \alpha\beta \cdot p_{n-2}$ lets you compute p_n without ever expanding α^n or β^n in full.

SUBSTITUTION AND POLYNOMIAL REDUCTION

PROBLEM 3.7. Given $x + \dfrac{1}{x} = 3$, find $x^5 + \dfrac{1}{x^5}$.

Technique. Define $p_n = x^n + x^{-n}$ and derive a recurrence. Each step walks us up by one unit of the exponent.

Discussion. Let $p_n = x^n + x^{-n}$. We are given $p_1 = 3$.

To find the recurrence, multiply p_n by p_1 and expand:

$$p_n \cdot p_1 = (x^n + x^{-n})(x + x^{-1}) = p_{n+1} + p_{n-1}.$$

Rearranging,

$$p_{n+1} = p_1 \cdot p_n - p_{n-1}.$$

With $p_1 = 3$, the recurrence becomes $p_{n+1} = 3p_n - p_{n-1}$.

Starting values:

$$p_0 = x^0 + x^0 = 2, \qquad p_1 = 3.$$

Now step up:

$$\begin{aligned} p_2 &= 3 \cdot 3 - 2 \;=\; 7, \\ p_3 &= 3 \cdot 7 - 3 \;=\; 18, \\ p_4 &= 3 \cdot 18 - 7 \;=\; 47, \\ p_5 &= 3 \cdot 47 - 18 \;=\; 123. \end{aligned}$$

□

What to take away. The substitution $p_n = x^n + x^{-n}$ converts a nonlinear-looking question into a linear recurrence. The same idea handles problems about $\cos(n\theta)$ (via Euler: $2\cos\theta = e^{i\theta} + e^{-i\theta}$), and underlies the Chebyshev polynomials. Whenever a problem features a sum of a quantity and its reciprocal, this substitution is usually the quickest route.

Problem 3.8. Suppose x is a real number satisfying $x^2 - x - 1 = 0$. Find integers a and b such that

$$x^5 \;=\; a + bx.$$

Technique. Use the given equation to rewrite x^2 in terms of x and 1. Successively reduce higher powers until x^5 is a linear combination of 1 and x.

Discussion. From the equation, $x^2 = x + 1$. We use this to reduce every higher power step by step:

$$\begin{aligned} x^2 &= x + 1, \\ x^3 &= x \cdot x^2 \;=\; x^2 + x \;=\; 2x + 1, \\ x^4 &= x \cdot x^3 \;=\; 2x^2 + x \;=\; 3x + 2, \\ x^5 &= x \cdot x^4 \;=\; 3x^2 + 2x \;=\; 5x + 3. \end{aligned}$$

So $x^5 = 3 + 5x$, that is, $(a, b) = (3, 5)$. □

What to take away. When x satisfies a polynomial equation of degree d, every polynomial in x (and therefore every power x^n) can be reduced to a polynomial of degree strictly less than d using the defining equation. The coefficients in the reduction of $x, x^2, x^3, x^4, x^5, \ldots$ for $x^2 = x + 1$ are $(0,1), (1,1), (1,2), (2,3), (3,5)$. These are consecutive pairs of Fibonacci numbers. That is no accident: the positive root of

$x^2 - x - 1 = 0$ is the golden ratio $\varphi = (1+\sqrt{5})/2$, and $\varphi^n = F_{n-1} + F_n\varphi$ in general.

TELESCOPING

PROBLEM 3.9. Compute the sum

$$\sum_{k=1}^{99} \frac{1}{k(k+1)}.$$

Technique. Partial fractions: split each term into a difference whose two halves cancel with neighbouring terms.

Discussion. Decompose $1/(k(k+1))$ into partial fractions. We want constants A, B with

$$\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{k+1}.$$

Clearing denominators: $1 = A(k+1) + Bk = (A+B)k + A$. Matching coefficients, $A = 1$ and $B = -1$, so

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

The sum therefore becomes

$$\sum_{k=1}^{99}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{99} - \frac{1}{100}\right).$$

Every $-\frac{1}{k+1}$ in one bracket cancels the $\frac{1}{k+1}$ that opens the next. Only the very first and the very last survive:

$$\sum_{k=1}^{99} \frac{1}{k(k+1)} = \frac{1}{1} - \frac{1}{100} = \frac{99}{100}.$$

□

What to take away. A sum telescopes whenever each term can be written as the difference of consecutive values of some sequence f. In that case,

$$\sum_{k=1}^{n} (f(k) - f(k+1)) = f(1) - f(n+1).$$

The partial-fraction decomposition $1/(k(k+1)) = 1/k - 1/(k+1)$ is the classical example; the same idea handles $\sum 1/(k(k+r))$ for any positive integer r, and many similar sums.

Problem 3.10. Compute the product

$$\prod_{k=2}^{100}\left(1-\frac{1}{k^2}\right).$$

Technique. Factor each term into a numerator and denominator whose pieces 'shift' from one term to the next, producing a product that collapses.

Discussion. Factor the kth term using $1 - 1/k^2 = (k-1)(k+1)/k^2$:

$$\prod_{k=2}^{100}\left(1-\frac{1}{k^2}\right) = \prod_{k=2}^{100}\frac{(k-1)(k+1)}{k^2}.$$

Splitting the product into three factors,

$$\prod_{k=2}^{100}\frac{(k-1)(k+1)}{k^2} = \frac{\prod_{k=2}^{100}(k-1)\cdot\prod_{k=2}^{100}(k+1)}{\prod_{k=2}^{100}k^2}.$$

Each factor in the numerator and denominator is a stretch of consecutive integers:

$$\begin{aligned}\prod_{k=2}^{100}(k-1) &= 1\cdot 2\cdot 3\cdots 99 = 99!,\\ \prod_{k=2}^{100}(k+1) &= 3\cdot 4\cdot 5\cdots 101 = \frac{101!}{1\cdot 2} = \frac{101!}{2},\\ \prod_{k=2}^{100}k &= 2\cdot 3\cdots 100 = 100!.\end{aligned}$$

Substitute and simplify:

$$\frac{99!\cdot(101!/2)}{(100!)^2} = \frac{99!}{100!}\cdot\frac{101!}{100!}\cdot\frac{1}{2} = \frac{1}{100}\cdot 101\cdot\frac{1}{2} = \frac{101}{200}.$$

☐

What to take away. Products telescope just like sums, once every term is factored into a piece that cancels across the product. The general result,

$$\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n},$$

follows from the same factorisation, and reveals that the infinite product $\prod_{k=2}^{\infty}(1 - 1/k^2) = 1/2$, a genuinely non-obvious fact about an infinite algebraic identity.

Chapter 4

Functions and graphs

Domain and range

Problem 4.1. Find the natural domain of

$$f(x) = \sqrt{x-1} + \sqrt{5-x}.$$

Technique. Each square root requires its radicand to be non-negative. The natural domain is the intersection of these requirements.

Discussion. For the expression to make sense we need both radicands to be ≥ 0:

$$\begin{aligned} x-1 &\geq 0 \quad \iff \quad x \geq 1, \\ 5-x &\geq 0 \quad \iff \quad x \leq 5. \end{aligned}$$

Both must hold simultaneously, so

$$x \in [1,5].$$

□

What to take away. For a function built by combining simpler pieces (square roots, logarithms, rational terms), the natural domain is the intersection of each piece's domain constraint. Miss one constraint and you have included points where the formula is nonsense.

Problem 4.2. Find the range of $f(x) = \sqrt{x^2+1} - x$ over real x.

Technique. Rationalise by multiplying by the conjugate, then read off the range of the resulting reciprocal form.

Discussion. Multiply top and bottom by $\sqrt{x^2+1} + x$. The numerator becomes $(x^2+1) - x^2 = 1$, leaving

$$f(x) = \sqrt{x^2+1} - x = \frac{1}{\sqrt{x^2+1} + x}.$$

Call the denominator $g(x) = \sqrt{x^2+1} + x$. Since $\sqrt{x^2+1} > |x| \geq -x$ for every real x, we have $g(x) > 0$, so $f(x) > 0$ always.

Check the ends:

- As $x \to +\infty$, $g(x) \to +\infty$, so $f(x) \to 0^+$.
- As $x \to -\infty$, $\sqrt{x^2+1} \to +\infty$ but $x \to -\infty$; more carefully, $g(x) = 1/f(x)$ and $f(x) \to +\infty$ (since $\sqrt{x^2+1} + |x| \to 2|x|$ while $\sqrt{x^2+1} - x$ becomes two large positives added). So $g(x) \to 0^+$.

Since g is continuous and takes arbitrarily large and arbitrarily small positive values, it takes every value in $(0, \infty)$. Hence $f = 1/g$ also takes every value in $(0, \infty)$.

$$\text{Range of } f = (0, \infty).$$

□

What to take away. The trick $\sqrt{A} - \sqrt{B} = (A - B)/(\sqrt{A} + \sqrt{B})$ is indispensable whenever you need to understand a difference of square roots. The rationalised form is usually tame where the original is subtle (here, the problematic $+\infty - \infty$ behaviour at $x \to -\infty$ becomes a clean 0^+ reciprocal).

Composition and inverses

Problem 4.3. Let $f(x) = x + 1$ and $g(x) = x^2$. Find all real x satisfying

$$f \circ g\,(x) = g \circ f\,(x).$$

Technique. Compose both ways explicitly, then solve the resulting equation.

Discussion. Compose:

$$\begin{aligned} f(g(x)) = f(x^2) &= x^2+1, \\ g(f(x)) = g(x+1) &= (x+1)^2 = x^2+2x+1. \end{aligned}$$

Set equal:

$$x^2+1 = x^2+2x+1 \implies 0 = 2x \implies x = 0.$$

☐

What to take away. Function composition is rarely commutative; $f \circ g = g \circ f$ is a special event that happens only at isolated points (or not at all). When you compose, simplify before setting things equal: often the higher-order terms cancel and what remains is a short linear equation, as here.

Problem 4.4. Find the inverse of $f(x) = \dfrac{x+1}{x-2}$ for $x \neq 2$, and state the domain and range of f^{-1}.

Technique. Write $y = f(x)$ and solve for x in terms of y. Remember that domain of f^{-1} is the range of f, and vice versa.

Discussion. Start with $y = (x+1)/(x-2)$ and solve for x:

$$y(x-2) = x+1, \qquad yx-2y = x+1, \qquad x(y-1) = 2y+1.$$

Provided $y \neq 1$,

$$x = \frac{2y+1}{y-1}.$$

Renaming the variable,

$$f^{-1}(x) = \frac{2x+1}{x-1}.$$

Domain and range of f^{-1}. The formula is defined for all $x \neq 1$, so the domain of f^{-1} is $\{x : x \neq 1\}$. The range of f^{-1} is the domain of f: $\{y : y \neq 2\}$.

A quick sanity check: the horizontal asymptote of $y = (x+1)/(x-2)$ is $y = 1$ (since numerator and denominator are both degree one with leading coefficient 1), so f never outputs 1. That matches the domain of f^{-1} excluding 1. □

What to take away. Inverting a rational function is an algebraic operation: solve for the old variable in terms of the new. Domain and range swap. The set of fractional linear functions $(ax+b)/(cx+d)$ with $ad-bc\neq 0$ is closed under inversion: the inverse is always another fractional linear function.

PARITY

PROBLEM 4.5. Every function f whose domain is symmetric about 0 (so that $f(-x)$ is defined wherever $f(x)$ is) can be written uniquely as $f=g+h$ with g even and h odd. Find this decomposition for $f(x)=\dfrac{1}{1-x}$, which lives on the symmetric domain $\mathbb{R}\setminus\{\pm 1\}$.

Technique. The universal formulas

$$g(x) = \tfrac{1}{2}(f(x)+f(-x)), \qquad h(x) = \tfrac{1}{2}(f(x)-f(-x))$$

give the decomposition for any f.

Discussion. Define g and h by the formulas above. Check:

- $g(-x)=\frac{1}{2}(f(-x)+f(x))=g(x)$, so g is even.
- $h(-x)=\frac{1}{2}(f(-x)-f(x))=-h(x)$, so h is odd.
- $g(x)+h(x)=f(x)$ by adding the two formulas.

Uniqueness: if $f=g_1+h_1=g_2+h_2$ with g_i even and h_i odd, then $g_1-g_2=h_2-h_1$. The left side is an even function; the right side is an odd function. The only function that is both even and odd is the zero function, so $g_1=g_2$ and $h_1=h_2$.

Apply to $f(x)=1/(1-x)$. Compute $f(-x)=1/(1+x)$. For the even part,

$$g(x) = \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right) = \frac{1}{2}\cdot\frac{2}{(1-x)(1+x)} = \frac{1}{1-x^2}.$$

For the odd part,

$$h(x) = \tfrac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right) = \tfrac{1}{2} \cdot \frac{2x}{(1-x)(1+x)} = \frac{x}{1-x^2}.$$

Quick check: $g(x) + h(x) = (1+x)/(1-x^2) = (1+x)/((1-x)(1+x)) = 1/(1-x)$.

$$g(x) = \frac{1}{1-x^2}, \qquad h(x) = \frac{x}{1-x^2}.$$

□

What to take away. Every function has a unique even-odd decomposition; the formulas $g = (f(x) + f(-x))/2$ and $h = (f(x) - f(-x))/2$ are universal. This is the simplest example of decomposing a function into symmetry components, an idea that reappears throughout analysis (Fourier series is a deeper version).

Problem 4.6. Is $f(x) = x\,|x|$ an even function, an odd function, or neither?

Technique. Compute $f(-x)$ and compare to $f(x)$ and $-f(x)$.

Discussion.

$$f(-x) = (-x) \cdot |-x| = -x \cdot |x| = -f(x).$$

So $f(-x) = -f(x)$: the function is odd. □

(A geometric check: for $x \geq 0, f(x) = x^2$; for $x \leq 0, f(x) = -x^2$. The graph is half a right-opening parabola on the positive side and half a left-opening parabola on the negative side, symmetric about the origin, just as an odd function should be.)

What to take away. To test parity, compute $f(-x)$ and compare. Most functions are neither even nor odd; those that are carry a visible symmetry (even: mirror across y-axis; odd: rotation by 180° about the origin). Products of even and odd follow the same rule as signs: even times even is even, odd times odd is even, even times odd is odd. Here x is odd and $|x|$ is even, so their product is odd.

MONOTONICITY AND SYMMETRY

PROBLEM 4.7. Show that $f(x) = x^3 + x$ is strictly increasing on $\mathbb{R}$.

Technique. Prove $f(a) > f(b)$ whenever $a > b$ by factoring $f(a) - f(b)$ and showing the second factor is always positive.

Discussion. Take any real $a > b$. Compute the difference and factor:

$$\begin{aligned} f(a) - f(b) &= (a^3 + a) - (b^3 + b) \\ &= (a^3 - b^3) + (a - b) \\ &= (a - b)(a^2 + ab + b^2) + (a - b) \\ &= (a - b)(a^2 + ab + b^2 + 1). \end{aligned}$$

We need to show the second factor is always positive. Multiplying by 2 and regrouping,

$$2(a^2 + ab + b^2) \;=\; (a + b)^2 + a^2 + b^2 \;\geq\; 0.$$

So $a^2 + ab + b^2 \geq 0$ for all real a, b, and therefore $a^2 + ab + b^2 + 1 \geq 1 > 0$.

Since $a > b$, the factor $(a - b) > 0$, and the product of two positive numbers is positive:

$$f(a) - f(b) \;>\; 0, \qquad \text{so} \qquad f(a) \;>\; f(b).$$

Thus f is strictly increasing. □

What to take away. Monotonicity can be established without calculus: factor the difference $f(a) - f(b)$ so that $(a - b)$ appears, and show the remaining factor has a definite sign. The identity $2(a^2 + ab + b^2) = (a + b)^2 + a^2 + b^2$ is the classical proof that the quadratic form $a^2 + ab + b^2$ is non-negative.

PROBLEM 4.8. Show that the graph of $y = \dfrac{x + 1}{x - 1}$ is symmetric about the point $(1, 1)$.

Technique. A graph is symmetric about the point (a, b) if and only if for every point (p, q) on the graph, the reflected point

$(2a-p, 2b-q)$ is also on the graph. This is equivalent to the identity $f(2a-x) = 2b - f(x)$ holding for all valid x.

Discussion. Call $f(x) = (x+1)/(x-1)$. We want to show $f(2-x) = 2 - f(x)$ for all $x \neq 1$.

Compute the left-hand side:

$$f(2-x) \;=\; \frac{(2-x)+1}{(2-x)-1} \;=\; \frac{3-x}{1-x}.$$

Compute the right-hand side:

$$2 - f(x) \;=\; \frac{2(x-1)-(x+1)}{x-1} \;=\; \frac{x-3}{x-1} \;=\; \frac{3-x}{1-x}.$$

The two sides match, so the identity $f(2-x) = 2 - f(x)$ holds for every valid x: the graph is symmetric about the point $(1, 1)$. □

What to take away. Point symmetry about (a, b) is the two-sided analogue of mirror symmetry: each point has a partner obtained by reflecting through (a, b). The condition $f(2a-x) = 2b - f(x)$ is the computational test. For any fractional linear function $f(x) = (px+q)/(rx+s)$, the graph is symmetric about the point where its two asymptotes intersect; here the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 1$, confirming the claim.

Graph transformations

Problem 4.9. The graph of $y = f(x)$ passes through the point $(2, 5)$. Through which point does the graph of

$$y \;=\; 3f(2x-4) \;+\; 1$$

pass?

Technique. Track the input into f and the output from f separately.

Discussion. We know $f(2) = 5$. The new graph uses f at the argument $2x-4$, so the new x-coordinate we care about is the one making the argument equal to 2:

$$2x - 4 \;=\; 2 \quad\Longrightarrow\quad x \;=\; 3.$$

At that x, the new y-coordinate is

$$y = 3f(2) + 1 = 3 \cdot 5 + 1 = 16.$$

The new graph therefore passes through $(3, 16)$. □

What to take away. A transformation $y = Af(B(x - c)) + D$ is applied in layers. Work from the inside out for the input ($x \to B(x - c)$), and from the outside in for the output ($f \to Af + D$). Horizontal transformations behave in reverse (a shift right by c inside the function corresponds to $x - c$ in the argument), while vertical transformations behave as they look.

Problem 4.10. The graph of $y = \sqrt{x}$ is reflected in the horizontal line $y = 1$. Write the equation of the reflected graph.

Technique. Reflection of a point (x, y) across the horizontal line $y = c$ sends it to $(x, 2c - y)$.

Discussion. A point on the original graph is $(x, \sqrt{x})$, valid for $x \geq 0$. Its reflection across the line $y = 1$ is

$$(x,\ 2 \cdot 1 - \sqrt{x}) = (x,\ 2 - \sqrt{x}).$$

So the reflected graph has equation

$$y = 2 - \sqrt{x}, \qquad x \geq 0.$$

□

What to take away. Reflection across the horizontal line $y = c$ replaces y by $2c - y$; the special case $c = 0$ (reflection across the x-axis) gives $y \to -y$. For vertical-line reflection $x = c$, replace x by $2c - x$. The point-reflection through (a, b) from the previous movement is the composition of both: $(x, y) \to (2a - x, 2b - y)$.

PERIODICITY FROM FUNCTIONAL IDENTITIES

PROBLEM 4.11. If $f\colon \mathbb{R} \to \mathbb{R}$ satisfies $f(x+2) = -f(x)$ for every real x, show that f has period 4.

Technique. Iterate the identity once: apply the given relation to $f(x+2)$ itself to reach $f(x+4)$.

Discussion. Apply the given identity with x replaced by $x+2$:

$$f((x+2)+2) \;=\; -f(x+2).$$

That is,

$$f(x+4) \;=\; -f(x+2).$$

Now substitute $f(x+2) = -f(x)$ from the original identity:

$$f(x+4) \;=\; -(-f(x)) \;=\; f(x).$$

This holds for every real x, so 4 is a period of f. □

What to take away. Many functional identities hide a period; the method is to iterate. The sign-flip identity $f(x+T/2) = -f(x)$ produces period T by the same two-step argument: applying it twice flips the sign twice. More generally, any identity that maps $f(x)$ to a second value that, under the same identity, maps back to $f(x)$, produces a period.

PROBLEM 4.12. If $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x+a) \;=\; \frac{1+f(x)}{1-f(x)}$$

for every real x, where $a > 0$ is a fixed constant, show that f has period $4a$.

Technique. Iterate the identity twice to discover an intermediate relation $f(x+2a) = -1/f(x)$, then once more to close the loop.

Discussion. First, the identity constrains f enough to make the divisions below legitimate. Its right side is undefined when $f(x) = 1$, so $f(x) \neq 1$ for every x. And $f(x) \neq 0$ for every

x: if $f(x_0) = 0$ for some x_0, then $f(x_0 + a) = (1+0)/(1-0) = 1$, and then $f(x_0 + 2a) = (1+1)/(1-1)$ would be undefined, contradicting that the identity holds for *all* real x. So f takes neither value 0 nor 1 anywhere.

Now let $F = f(x)$ for brevity. Apply the given identity to $f(x+a)$ itself:

$$f(x+2a) = \frac{1+f(x+a)}{1-f(x+a)} = \frac{1+(1+F)/(1-F)}{1-(1+F)/(1-F)}.$$

Simplify the numerator and denominator separately:

$$1 + \frac{1+F}{1-F} = \frac{(1-F)+(1+F)}{1-F} = \frac{2}{1-F},$$

$$1 - \frac{1+F}{1-F} = \frac{(1-F)-(1+F)}{1-F} = \frac{-2F}{1-F}.$$

Dividing,

$$f(x+2a) = \frac{2/(1-F)}{-2F/(1-F)} = -\frac{1}{F} = -\frac{1}{f(x)}.$$

Now iterate once more. Apply this intermediate identity to $f(x+2a)$:

$$f(x+4a) = -\frac{1}{f(x+2a)} = -\frac{1}{-1/f(x)} = f(x).$$

So f has period $4a$. □

What to take away. When the functional identity is non-linear, iteration can traverse several distinct relations before returning. Here the four steps are

$$f(x) \mapsto \frac{1+f(x)}{1-f(x)} \mapsto -\frac{1}{f(x)} \mapsto \frac{f(x)-1}{f(x)+1} \mapsto f(x).$$

The map $F \mapsto (1+F)/(1-F)$ has order 4 as a transformation; the function f inherits that order as a period.

ITERATION AND FIXED POINTS

PROBLEM 4.13. For $f(x) = \dfrac{x}{1+x}$, let f_n denote the n-fold iterate of f, that is $f_n = f \circ f \circ \cdots \circ f$ with n copies of f $(n \geq 1)$. Find a closed form for $f_n(x)$.

Technique. Compute the first few iterates, spot the pattern, verify by induction.

Discussion. Compute step by step. The first iterate is $f_1(x) = x/(1+x)$. For the second,

$$f_2(x) = f\left(\frac{x}{1+x}\right) = \frac{x/(1+x)}{(1+2x)/(1+x)} = \frac{x}{1+2x}.$$

For the third,

$$f_3(x) = f\left(\frac{x}{1+2x}\right) = \frac{x/(1+2x)}{(1+3x)/(1+2x)} = \frac{x}{1+3x}.$$

The pattern is clear:

$$f_n(x) = \frac{x}{1+nx},$$

valid wherever the whole chain of compositions is defined. Since f has its pole at -1 and the k-th value in the chain is $f_k(x) = x/(1+kx)$, which equals -1 exactly when $x = -1/(k+1)$, the composition f_n is defined for every real x except $x = -1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}$. Verify by induction. The base case $n = 1$ matches. For the inductive step, assume $f_n(x) = x/(1+nx)$. Then

$$f_{n+1}(x) = \frac{f_n(x)}{1+f_n(x)} = \frac{x/(1+nx)}{(1+(n+1)x)/(1+nx)},$$

which simplifies to $x/(1+(n+1)x)$, closing the induction. □

What to take away. Iterating a rational function often produces a family of the same form, indexed by n. Spot the pattern from the first few iterates, then prove by induction. For this particular f, the substitution $g = 1/f$ converts iteration into arithmetic: $1/f_n(x) = 1/x + n$ is simply an arithmetic progression in n, which explains the pattern.

Problem 4.14. Show that $f(x) = \dfrac{1+x}{1-x}$ satisfies $f^4(x) = x$ for every x for which the fourth iterate is defined, namely $x \in \mathbb{R} \setminus \{-1, 0, 1\}$.

Technique. Compute $f^2(x)$; it simplifies dramatically. Then $f^4 = f^2 \circ f^2$ is even simpler.

Discussion. Compute $f^2(x) = f(f(x))$:

$$f^2(x) = \frac{1+f(x)}{1-f(x)} = \frac{1+(1+x)/(1-x)}{1-(1+x)/(1-x)}.$$

Simplify numerator and denominator:

$$1 + \frac{1+x}{1-x} = \frac{2}{1-x}, \qquad 1 - \frac{1+x}{1-x} = \frac{-2x}{1-x}.$$

Divide:

$$f^2(x) = \frac{2/(1-x)}{-2x/(1-x)} = -\frac{1}{x}.$$

Now apply f^2 to itself:

$$f^4(x) = f^2(f^2(x)) = f^2\left(-\frac{1}{x}\right) = -\frac{1}{-1/x} = x,$$

valid for every x in the domain of f^4, which is $\mathbb{R} \setminus \{-1, 0, 1\}$. The exclusions are forced: f itself is undefined at 1; the intermediate $f^2(x) = -1/x$ is undefined at 0; and from $x = -1$ the chain runs into the pole at the third step, since $f(-1) = 0$, $f^2(-1) = 1$, and $f^3(-1) = f(1)$ is undefined. □

What to take away. Composing a fractional linear function with itself is another fractional linear function. In this case, the iterates cycle through four distinct maps

$$x, \qquad \frac{1+x}{1-x}, \qquad -\frac{1}{x}, \qquad \frac{x-1}{x+1}, \qquad x, \qquad \ldots$$

before returning. This is the same 4-cycle that appeared in the earlier periodicity problem, which is not a coincidence: any function f satisfying $f(x+a) = (1+f(x))/(1-f(x))$ inherits the period of the underlying map $F \mapsto (1+F)/(1-F)$.

PIECEWISE FUNCTIONS AND THE FLOOR

PROBLEM 4.15. Write $f(x) = |x-1| + |x+1|$ as a piecewise formula. State the range of f and identify the set of x where f attains its minimum.

Technique. Split the real line at the sign-change points $x = -1$ and $x = 1$, and write f as a linear expression on each of the three resulting intervals.

Discussion. The two absolute values change sign at $x = -1$ (for $|x+1|$) and $x = 1$ (for $|x-1|$). This divides the real line into three intervals.

Case $x \le -1$. Both $x - 1 < 0$ and $x + 1 \le 0$. So $|x-1| = 1 - x$ and $|x+1| = -1 - x$. Sum:

$$f(x) \;=\; (1-x) + (-1-x) \;=\; -2x.$$

Case $-1 \le x \le 1$. Now $x-1 \le 0$ but $x+1 \ge 0$. So $|x-1| = 1-x$ and $|x+1| = x+1$. Sum:

$$f(x) \;=\; (1-x) + (x+1) \;=\; 2.$$

Case $x \ge 1$. Both non-negative:

$$f(x) \;=\; (x-1) + (x+1) \;=\; 2x.$$

Piecewise formula:

$$f(x) \;=\; \begin{cases} -2x, & x \le -1, \\ 2, & -1 \le x \le 1, \\ 2x, & x \ge 1. \end{cases}$$

Range. On $x \le -1$, $-2x \ge 2$; on $-1 \le x \le 1$, $f = 2$; on $x \ge 1$, $2x \ge 2$. Every output is at least 2, and every value ≥ 2 is attained. So

$$\text{Range} \;=\; [2, \infty).$$

Minimum. The minimum value is 2, attained for every x in the interval $[-1, 1]$. □

What to take away. A sum of absolute values $|x-a_1|+|x-a_2|+\cdots+|x-a_n|$ has a piecewise-linear graph: its slope changes at each a_k and increases by 2 there. The minimum occurs at the median of the a_k's (or the middle interval, if n is even). Here $n = 2$, median $=$ $(-1,1)$-interval, giving the flat bottom on $[-1,1]$.

Problem 4.16. Find all real numbers x satisfying

$$\lfloor x \rfloor + \lfloor -x \rfloor = 0.$$

Technique. Split into two cases: x integer, x not integer. Use the identity $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ for non-integer x.

Discussion.

Case x is an integer. Then $\lfloor x \rfloor = x$ and $\lfloor -x \rfloor = -x$. Their sum is $x + (-x) = 0$, and the equation holds.

Case x is not an integer. Write $x = n + \theta$ where $n = \lfloor x \rfloor$ is an integer and $0 < \theta < 1$ is the fractional part. Then

$$-x = -n - \theta,$$

and since $-\theta$ lies in $(-1,0)$, the value $-n-\theta$ lies in $(-n-1,-n)$. Therefore

$$\lfloor -x \rfloor = -n - 1.$$

The sum is $\lfloor x \rfloor + \lfloor -x \rfloor = n + (-n-1) = -1$, not zero.

So the equation holds if and only if x is an integer.

$$x \in \mathbb{Z}.$$

□

What to take away. The two identities

$$\lfloor -x \rfloor = -\lfloor x \rfloor \quad (x \in \mathbb{Z}), \qquad \lfloor -x \rfloor = -\lfloor x \rfloor - 1 \quad (x \notin \mathbb{Z})$$

are the workhorses for problems involving floor and negation. The sum $\lfloor x \rfloor + \lfloor -x \rfloor$ is therefore a crude integer detector: 0 at integers, -1 everywhere else.

CHAPTER 5

Inequalities and absolute value

POLYNOMIAL INEQUALITIES VIA THE SIGN CHART

PROBLEM 5.1. Solve $(x-1)(x+2)(x-3) \geq 0$.

Technique. The sign chart. A product of linear factors changes sign at each of its zeros; on each open interval between consecutive zeros, the sign of the product is constant, and it flips as you cross a zero.

Discussion. The three zeros of the product are $x = -2, 1, 3$. They divide the real line into four open intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$. Pick a test point in each and evaluate the sign of the three factors.

interval	$x-1$	$x+2$	$x-3$	product
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < 1$	$-$	$+$	$-$	$+$
$1 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

The product is ≥ 0 on the two intervals where the sign is $+$, together with the three zeros themselves (where the product equals zero). So:

$$x \in [-2, 1] \cup [3, \infty).$$

□

What to take away. For a product of distinct linear factors, the sign alternates as you cross each zero. You only need to check one test point (or, more quickly, read off the sign from the leading behaviour at $x \to +\infty$, then alternate leftward). Close-bracketed endpoints are included when the inequality is weak ($\geq$ or $\leq$), excluded when strict.

Problem 5.2. Solve $x^4 - 5x^2 + 4 > 0$.

Technique. Substitution $u = x^2$ turns a biquadratic expression into a quadratic in u; factor, solve the u-inequality, then back-substitute.

Discussion. Substitute $u = x^2$, so $x^4 = u^2$ and the inequality becomes

$$u^2 - 5u + 4 \; > \; 0.$$

Factor: $u^2-5u+4 = (u-1)(u-4)$. So we need $(u-1)(u-4) > 0$, which holds when $u < 1$ or $u > 4$.

Now back-substitute $u = x^2$:

- $x^2 < 1$ gives $-1 < x < 1$.
- $x^2 > 4$ gives $x < -2$ or $x > 2$.

Taking the union:

$$x \in (-\infty, -2) \cup (-1, 1) \cup (2, \infty).$$

□

What to take away. A biquadratic (a polynomial with only even powers of x) always reduces to a quadratic in $u = x^2$. Once the u-inequality is solved, back-substitute carefully: $u < c$ corresponds to $|x| < \sqrt{c}$, while $u > c$ corresponds to $|x| > \sqrt{c}$ (assuming $c \geq 0$).

RATIONAL INEQUALITIES

PROBLEM 5.3. Solve $\frac{x^2-4}{x-1} \le 0$.

Technique. Factor numerator and denominator. Track the sign of each linear factor across the real line, excluding the points where the denominator vanishes.

Discussion. Factor the numerator: $x^2-4=(x-2)(x+2)$. So the inequality becomes

$$\frac{(x-2)(x+2)}{x-1} \le 0, \qquad x \neq 1.$$

The critical points are $x=-2,1,2$. Sign chart:

interval	$x-2$	$x+2$	$x-1$	ratio
$x<-2$	$-$	$-$	$-$	$-$
$-2<x<1$	$-$	$+$	$-$	$+$
$1<x<2$	$-$	$+$	$+$	$-$
$x>2$	$+$	$+$	$+$	$+$

The ratio is ≤ 0 on the two intervals where it is negative, plus the zeros of the numerator: $x=-2$ and $x=2$. Include the zeros; exclude $x=1$ (denominator zero):

$$x \in (-\infty,-2] \cup (1,2].$$

□

What to take away. For a rational inequality, the sign-chart method works just as for polynomials, with one crucial twist: the points where the denominator vanishes are excluded from the solution. Missing that exclusion is the most common error.

PROBLEM 5.4. Solve $\frac{1}{x-1}+\frac{1}{x+1} > 0$.

Technique. Combine the two fractions over a common denominator so the left-hand side becomes a single ratio. Then sign-chart.

Discussion. Combine:

$$\frac{1}{x-1}+\frac{1}{x+1} = \frac{(x+1)+(x-1)}{(x-1)(x+1)} = \frac{2x}{x^2-1}.$$

So the inequality becomes

$$\frac{2x}{(x-1)(x+1)} \;>\; 0, \qquad x \neq \pm 1.$$

Critical points $x = -1, 0, 1$. Sign chart:

interval	$2x$	$x-1$	$x+1$	ratio
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

The strict inequality > 0 is satisfied on the $+$ intervals, not at any zero:

$$x \in (-1, 0) \cup (1, \infty).$$

□

What to take away. Combine fractions before sign-charting, otherwise you will miss the contribution that each term makes to the sign of the sum. A common trap: imagine the two terms cancelling at $x = 0$ and concluding wrongly that $x = 0$ is a solution. The combined form reveals that $x = 0$ makes the ratio zero, not positive.

ABSOLUTE-VALUE INEQUALITIES

PROBLEM 5.5. Solve $|x-1| < |x+2|$ for real x.

Technique. Both sides are non-negative, so the inequality $|A| < |B|$ is equivalent to $A^2 < B^2$. Square, cancel, and solve what remains.

Discussion. Both sides of $|x-1| < |x+2|$ are non-negative, so the inequality is equivalent to

$$(x-1)^2 \;<\; (x+2)^2.$$

Expand:

$$x^2 - 2x + 1 \;<\; x^2 + 4x + 4.$$

The x^2 cancels:

$$-6x < 3 \qquad \iff \qquad x > -\tfrac{1}{2}.$$

$$x \in \left(-\tfrac{1}{2}, \infty\right).$$

□

What to take away. Inequalities between two absolute values are always handled by squaring, since both sides are non-negative (so squaring preserves the inequality). After squaring, the x^2 terms usually cancel, leaving a linear condition. Geometrically, $|x-1| < |x+2|$ says "x is closer to 1 than to -2", which is exactly the condition $x > (1+(-2))/2 = -1/2$: to the right of the midpoint.

Problem 5.6. Solve $|x-1| + |x-4| \leq 5$.

Technique. Split at each sign-change point of the absolute values ($x = 1$ and $x = 4$). On each of the three resulting intervals, the expression is linear; solve on each and union the pieces.

Discussion. The sign-change points are $x = 1$ and $x = 4$, dividing the real line into three intervals.

Case $x \leq 1$. Here $|x-1| = 1-x$ and $|x-4| = 4-x$. Sum: $(1-x)+(4-x) = 5-2x$. Condition:

$$5-2x \leq 5 \qquad \iff \qquad x \geq 0.$$

Combined with the case range: $0 \leq x \leq 1$.

Case $1 \leq x \leq 4$. Here $|x-1| = x-1$ and $|x-4| = 4-x$. Sum: $(x-1)+(4-x) = 3$. Condition: $3 \leq 5$, true for every x in this range. So $1 \leq x \leq 4$.

Case $x \geq 4$. Here $|x-1| = x-1$ and $|x-4| = x-4$. Sum: $2x-5$. Condition:

$$2x-5 \leq 5 \qquad \iff \qquad x \leq 5.$$

Combined: $4 \leq x \leq 5$.

Union the three pieces:

$$x \in [0, 5].$$

□

What to take away. The sum $|x-a|+|x-b|$ is piecewise linear: it is constant on the interval $[a, b]$ (equal to $|a-b|$) and grows linearly as you move away from that interval. Inequalities involving such sums split into three cases (for two absolute values), $n+1$ in general. Check each candidate solution lies in the range of the case that produced it.

INEQUALITIES INVOLVING SQUARE ROOTS

PROBLEM 5.7. Solve $\sqrt{x+1} > x-1$.

Technique. The left side is non-negative (when defined). Split on the sign of the right side: if it is negative, the inequality holds automatically; if non-negative, both sides are non-negative and can be squared.

Discussion. Domain: the square root requires $x+1 \geq 0$, so $x \geq -1$.

Case $x-1<0$, that is $x<1$. The left side is ≥ 0 and the right side is <0, so the inequality is automatically true. Intersect with the domain: $-1 \leq x < 1$.

Case $x-1 \geq 0$, that is $x \geq 1$. Both sides are non-negative; squaring preserves the inequality:

$$x+1 \;>\; (x-1)^2 \;=\; x^2-2x+1.$$

Rearrange:

$$0 \;>\; x^2-3x \;=\; x(x-3).$$

The product $x(x-3)$ is negative when $0<x<3$. Intersect with the case range $x \geq 1$: $1 \leq x < 3$.

Union the two cases: $[-1, 1) \cup [1, 3) = [-1, 3)$.

$$x \in [-1, 3).$$

□

What to take away. A square root is always non-negative (where defined), so the inequality $\sqrt{\text{thing}} \;>\;$ other is automatic whenever the right-hand side is negative. Squaring is only legitimate when both sides are non-negative; otherwise it can introduce spurious solutions. The case split on the sign of the right-hand side is the discipline that keeps the solution honest.

Problem 5.8. Solve $\sqrt{2x+3} \geq x$.

Technique. Split on the sign of the right side; in the non-negative case, square.

Discussion. Domain: $2x + 3 \geq 0$, so $x \geq -\frac{3}{2}$.

Case $x \leq 0$. The left side is ≥ 0 and the right side is ≤ 0, so the inequality is automatically true. Combined with the domain: $-\frac{3}{2} \leq x \leq 0$.

Case $x > 0$. Both sides non-negative; square:

$$2x + 3 \ \geq \ x^2.$$

Rearrange:

$$x^2 - 2x - 3 \ \leq \ 0 \qquad \Longleftrightarrow \qquad (x-3)(x+1) \ \leq \ 0.$$

This holds for $-1 \leq x \leq 3$. Intersect with $x > 0$: $0 < x \leq 3$.

Union: $\left[-\frac{3}{2}, 0\right] \cup (0, 3] = \left[-\frac{3}{2}, 3\right]$.

$$x \in \left[-\frac{3}{2}, 3\right].$$

□

What to take away. Every radical inequality of the form $\sqrt{A} \geq B$ splits on the sign of B. If $B \leq 0$, the inequality holds trivially (as long as the radical is defined). If $B > 0$, square. Unlike the previous problem, here both cases contribute non-trivially to the final answer, giving a single closed interval.

PARAMETER PROBLEMS AND SYSTEMS

Problem 5.9. Find all real values of a for which the inequality $|x| + |x-3| \leq a$ has at least one real solution.

Technique. An inequality $f(x) \leq a$ has a solution if and only if $a \geq \min_x f(x)$. Find the minimum of the left-hand side by the piecewise analysis from the previous movement.

Discussion. Define $f(x) = |x|+|x-3|$. The sign-change points are $x = 0$ and $x = 3$, giving three cases:

- $x \leq 0$: $f(x) = -x + (3 - x) = 3 - 2x$. This is ≥ 3, with equality at $x = 0$.
- $0 \leq x \leq 3$: $f(x) = x + (3 - x) = 3$. Constant.
- $x \geq 3$: $f(x) = x + (x - 3) = 2x - 3$. This is ≥ 3, with equality at $x = 3$.

So the minimum of f is 3, attained at every $x \in [0, 3]$.

The inequality $f(x) \leq a$ has at least one solution if and only if a is at least as large as the minimum of f:

$$a \geq 3.$$

□

What to take away. The phrase "has at least one solution" shifts the question from solving the inequality to finding the minimum of one side. A parameter inequality like this is really a question about the extrema of the other side: $f(x) \leq a$ *always* solvable when $a \geq \min f$, *never* solvable when $a < \min f$. The same logic, flipped, handles "for all" parameter questions.

Problem 5.10. Find all real x satisfying both

$$(x-1)(x-2) > 0 \qquad \text{and} \qquad (x-1)(x-4) < 0.$$

Technique. Solve each inequality separately by sign chart, then intersect the solution sets.

Discussion. Sign chart for the first inequality $(x-1)(x-2) > 0$:

- $x < 1$: $(-)(-) = +$. Satisfies.
- $1 < x < 2$: $(+)(-) = -$. Fails.
- $x > 2$: $(+)(+) = +$. Satisfies.

So the first inequality gives $x < 1$ or $x > 2$.

Sign chart for the second inequality $(x-1)(x-4) < 0$:

- $x < 1$: $(-)(-) = +$. Fails.
- $1 < x < 4$: $(+)(-) = -$. Satisfies.
- $x > 4$: $(+)(+) = +$. Fails.

So the second inequality gives $1 < x < 4$.

Intersect the two solution sets:

- $x < 1$ intersected with $1 < x < 4$: empty.
- $x > 2$ intersected with $1 < x < 4$: $2 < x < 4$.

$$x \in (2, 4).$$

$\square$

What to take away. A system of inequalities (joined by "and") is solved by intersecting the solution sets of each inequality separately. Drawing the two sign charts on adjacent number lines, then marking the overlap, is often the fastest route to the answer. If the system is joined by "or", take the union instead.

CHAPTER 6

Mathematical induction

✠

WEAK INDUCTION

PROBLEM 6.1. Prove that for every positive integer n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 \;=\; \frac{n(n+1)(2n+1)}{6}.$$

Technique. Weak induction: verify the claim at $n = 1$; then show that if it holds at n, it also holds at $n + 1$.

Discussion. The base case $n = 1$: the left side is $1^2 = 1$, the right side is $\frac{1\cdot 2\cdot 3}{6} = 1$. Equal.

For the induction step, suppose the identity holds at n, so that $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Add $(n+1)^2$ to both sides:

$$1^2 + \cdots + n^2 + (n+1)^2 \;=\; \frac{n(n+1)(2n+1)}{6} + (n+1)^2.$$

The right-hand side becomes, after factoring $(n+1)$ out of the numerator,

$$\frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \;=\; \frac{(n+1)(2n^2+7n+6)}{6}.$$

A second factorisation, $2n^2 + 7n + 6 = (n+2)(2n+3)$, gives

$$\frac{(n+1)(n+2)(2n+3)}{6},$$

which is exactly the stated formula with n replaced by $n + 1$. The step goes through, and the identity holds for every n. □

What to take away. The inductive shell is just bookkeeping: base case, then step. The real work is the algebraic identity $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$. Every proof-by-induction of a sum formula reduces to one line of polynomial algebra.

Problem 6.2. Prove that for every positive integer n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Technique. Induction on n, with the step reducing to the identity $T_n^2 + (n+1)^3 = T_{n+1}^2$, where $T_n = n(n+1)/2$ is the n-th triangular number.

Discussion. Base $n = 1$: left side $1^3 = 1$, right side $[\frac{1\cdot 2}{2}]^2 = 1$. Equal.

For the step, assume the identity holds at n. Adding $(n+1)^3$ to both sides,

$$1^3 + \cdots + (n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3.$$

Pulling out $(n+1)^2$ from the right-hand side gives $(n+1)^2[\frac{n^2}{4} + (n+1)]$. The bracket simplifies to $(n^2+4n+4)/4 = (n+2)^2/4$, so the expression equals $[\frac{(n+1)(n+2)}{2}]^2$ as required. □

What to take away. The right-hand side $[\frac{n(n+1)}{2}]^2$ is the square of the n-th triangular number. So the sum of the first n cubes equals the square of the sum of the first n integers, a much-loved identity sometimes called Nicomachus's theorem. The induction proves it but does not explain why the identity is so clean; a combinatorial proof ("count pairs of pairs of integers") eventually will.

Problem 6.3. Prove that for every positive integer n, $n^7 - n$ is divisible by 7.

Technique. Induction on n; the step rearranges $(n+1)^7 - (n+1)$ as $(n^7 - n)$ plus a manifestly-divisible correction.

Discussion. Base $n = 1$: $1^7 - 1 = 0$, divisible by 7.

For the step, assume $7 \mid n^7 - n$. Consider

$$(n+1)^7 - (n+1) - (n^7 - n) \;=\; (n+1)^7 - n^7 - 1.$$

Expand $(n+1)^7$ by the binomial theorem:

$$(n+1)^7 \;=\; n^7 + \binom{7}{1}n^6 + \binom{7}{2}n^5 + \cdots + \binom{7}{6}n + 1.$$

Subtracting $n^7 + 1$ leaves

$$(n+1)^7 - n^7 - 1 \;=\; \binom{7}{1}n^6 + \binom{7}{2}n^5 + \cdots + \binom{7}{6}n.$$

Every coefficient $\binom{7}{k}$ for $1 \le k \le 6$ is divisible by 7, because 7 is prime and the factor 7 in the numerator of $\binom{7}{k} = \frac{7!}{k!(7-k)!}$ cannot be cancelled by any factor in the denominator. So the whole expression is a multiple of 7, and therefore $(n+1)^7 - (n+1) = (n^7 - n) + 7 \cdot (\text{integer})$ is divisible by 7. □

What to take away. The fact that $p \mid \binom{p}{k}$ for every prime p and every $1 \le k \le p - 1$ is the engine here. It generalises to Fermat's little theorem, the statement that $p \mid n^p - n$ for every prime p, which is proved the same way. The induction step on n trades a statement about $n + 1$ for a statement about n plus a binomial correction.

Problem 6.4. Prove that for every non-negative integer n, $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Technique. Induction on n; the step splits $11^{n+3} + 12^{2n+3}$ into 11 times the n-case plus $133 \cdot 12^{2n+1}$.

Discussion. Base $n = 0$: $11^2 + 12 = 121 + 12 = 133$. Divisible by 133.

For the step, assume $133 \mid 11^{n+2} + 12^{2n+1}$. Compute

$$11^{n+3} + 12^{2n+3} \;=\; 11 \cdot 11^{n+2} + 144 \cdot 12^{2n+1},$$

and rewrite $144 = 11 + 133$:

$$11^{n+3} + 12^{2n+3} \;=\; 11 \cdot 11^{n+2} + 11 \cdot 12^{2n+1} + 133 \cdot 12^{2n+1}.$$

Regroup:

$$11^{n+3} + 12^{2n+3} \;=\; 11(11^{n+2} + 12^{2n+1}) + 133 \cdot 12^{2n+1}.$$

The first term is 11 times a multiple of 133 (by hypothesis); the second is 133 times an integer. Both are divisible by 133, so their sum is too. □

What to take away. The trick was to rewrite the coefficient $144 = 12^2$ as $11 + 133$. The 133 is what we want; the 11 lets us reuse the inductive hypothesis on the $11^{n+2} + 12^{2n+1}$ block. When two bases differ by a fixed amount (here $12 - 11 = 1$, which lifts to $12^2 - 11 = 133$), an algebraic re-writing often makes an induction work.

STRONG INDUCTION

PROBLEM 6.5. Prove that every integer $n \geq 2$ can be written as a product of prime numbers.

Technique. Strong induction: assume the claim for every integer in $\{2, 3, \ldots, n\}$, and use it to prove the claim for $n + 1$.

Discussion. Base $n = 2$: 2 is itself prime, and the single-prime product $2 = 2$ is a valid factorisation.

For the step, let $n \geq 2$ and suppose every integer in $\{2, 3, \ldots, n\}$ is a product of primes. Consider $n + 1$. Two cases.

- If $n + 1$ is prime, then $n + 1$ itself is a one-factor prime product. Done.
- If $n + 1$ is composite, write $n + 1 = ab$ with $2 \leq a, b \leq n$. By the inductive hypothesis (applied separately to a and to b, both of which lie in $\{2, \ldots, n\}$), each of a and b is a product of primes. Concatenating the two prime products gives a prime product for $n + 1$.

Either way, $n + 1$ is a product of primes. □

What to take away. Weak induction would fail here: when $n + 1$ splits as ab, the factors a and b can be anywhere in $\{2, \ldots, n\}$, not just at n. Strong induction, which assumes every smaller case rather than only the immediately preceding one, matches the shape of the factorisation directly. The rule of thumb: whenever a problem naturally produces a smaller object of unpredictable size, reach for strong induction.

Problem 6.6. The sequence $v_0, v_1, v_2, \ldots$ is defined by $v_0 = 2$, $v_1 = 3$, and the recurrence $v_{n+1} = 3v_n - 2v_{n-1}$ for $n \geq 1$. Prove that $v_n = 2^n + 1$ for every $n \geq 0$.

Technique. Strong induction on n: verify the claim at $n = 0$ and $n = 1$, then show it at $n+1$ assuming it at both n and $n-1$.

Discussion. Base cases: $v_0 = 2 = 2^0 + 1$, and $v_1 = 3 = 2^1 + 1$. Both hold.

For the step, let $n \geq 1$ and assume $v_n = 2^n + 1$ and $v_{n-1} = 2^{n-1} + 1$. Substitute into the recurrence:

$$v_{n+1} \;=\; 3v_n - 2v_{n-1} \;=\; 3(2^n + 1) - 2(2^{n-1} + 1).$$

Expanding and gathering,

$$v_{n+1} \;=\; 3 \cdot 2^n + 3 - 2^n - 2 \;=\; 2 \cdot 2^n + 1 \;=\; 2^{n+1} + 1,$$

which is the formula at $n + 1$. □

What to take away. A two-step recurrence needs two base cases and a step that assumes both prior cases. Strong induction is the natural tool. Note also the arithmetic of the step: once the recurrence is written out with $2^n + 1$ substituted for v_n, the algebra is linear in 2^n and collapses immediately. In later chapters the same pattern recovers closed forms for every linear recurrence with constant coefficients.

FORWARD-BACKWARD INDUCTION

Problem 6.7. Prove that for every integer $n \geq 2$ and every tuple of positive reals $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \;\geq\; \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Technique. Cauchy's forward-backward induction: prove the inequality for $n = 2$; double the value of n by induction (forward); then show that the inequality at $n+1$ forces the inequality at n (backward). These two moves together reach every integer ≥ 2.

Discussion. The $n = 2$ case. For positive a, b,

$$\left(\frac{a+b}{2}\right)^2 - ab = \frac{(a-b)^2}{4} \geq 0,$$

with equality iff $a = b$. Taking square roots gives $\frac{a+b}{2} \geq \sqrt{ab}$.

Forward: AM-GM at n implies AM-GM at $2n$. Let $y_1, \dots, y_{2n}$ be positive. Split into two halves of size n. By the hypothesis,

$$\frac{y_1 + \cdots + y_n}{n} \geq \sqrt[n]{y_1 \cdots y_n},$$

and similarly $\frac{y_{n+1}+\cdots+y_{2n}}{n} \geq \sqrt[n]{y_{n+1} \cdots y_{2n}}$. Apply the $n = 2$ case to the two right-hand sides. Writing $P = \sqrt[n]{y_1 \cdots y_n}$ and $Q = \sqrt[n]{y_{n+1} \cdots y_{2n}}$,

$$\frac{P+Q}{2} \geq \sqrt{PQ} = \sqrt[2n]{y_1 \cdots y_{2n}}.$$

The average of the two n-term arithmetic means is $\frac{y_1+\cdots+y_{2n}}{2n}$, so chaining the inequalities gives AM-GM at $2n$.

Backward: AM-GM at $n+1$ implies AM-GM at n. Let $x_1, \dots, x_n$ be positive, and set

$$A = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Apply AM-GM at $n+1$ to the tuple $(x_1, \dots, x_n, A)$:

$$\frac{x_1 + \cdots + x_n + A}{n+1} \geq \sqrt[n+1]{x_1 \cdots x_n \cdot A}.$$

The left side equals A, so $A \geq \sqrt[n+1]{x_1 \cdots x_n \cdot A}$. Raising both sides to the $(n+1)$-th power and dividing by A,

$$A^{n+1} \geq x_1 \cdots x_n \cdot A \qquad \text{and so} \qquad A^n \geq x_1 \cdots x_n,$$

giving $A \geq \sqrt[n]{x_1 \cdots x_n}$, which is AM-GM at n.

Combining. Starting from $n = 2$, the forward step proves AM-GM for all powers of 2. The backward step lets us fill in every integer between consecutive powers of 2: from AM-GM at 2^k we descend to $2^k - 1, 2^k - 2, \ldots$, all the way down to the previous power. Every integer ≥ 2 is reached. Equality tracks through each step and holds only when all arguments coincide. □

What to take away. Ordinary induction climbs one step at a time. Cauchy's forward-backward induction climbs in leaps (doubling) and descends in small steps. The crucial insight in the backward step is to use the arithmetic mean A itself as the auxiliary entry; this is what collapses the larger inequality down to the smaller one. This is the classical proof of AM-GM, and the template, once seen, applies to many statements indexed by an integer n for which the step at $2n$ is easy and the step at n is hard. We meet this inequality again in Chapter 14 (Problem 14.1), where it returns not as an induction exercise but as the workhorse of the classical inequalities; the proof is recalled there so that the chapter stands on its own.

Well-ordering and infinite descent

Problem 6.8. Prove that there is no pair of positive integers (n, m) with $n^2 = 2m^2$. (Equivalently, $\sqrt{2}$ is irrational.)

Technique. Infinite descent, a flavour of proof by contradiction: assume a positive-integer solution exists, choose the one with smallest n, and construct a solution with a strictly smaller n.

Discussion. Suppose for contradiction that positive-integer solutions (n, m) to $n^2 = 2m^2$ exist. Among all such solutions, pick one whose first coordinate n is the least possible; this is legitimate because the set of first coordinates is a non-empty subset of the positive integers, which is well-ordered.

From $n^2 = 2m^2$, the right-hand side is even, so n^2 is even, so n is even. Write $n = 2n'$ with $n' \geq 1$. Substituting,

$$4n'^2 = 2m^2 \qquad \Longleftrightarrow \qquad m^2 = 2n'^2.$$

So (m, n') is also a positive-integer solution. We have $m^2 = n^2/2 < n^2$, so $m < n$. But (m, n') has first coordinate m, which is strictly smaller than n: contradiction with the minimality of n. □

What to take away. Descent is proof by contradiction in disguise: pretend a counterexample exists, construct a smaller one, and appeal to the fact that the positive integers cannot decrease forever. The same skeleton underlies Fermat's original argument that $x^4 + y^4 = z^2$ has no positive-integer solutions, which was the birth of the descent technique in 1637.

Problem 6.9. Let $a \geq b \geq 1$ be integers. The Euclidean algorithm transforms the pair (a, b) into $(b, a \bmod b)$ repeatedly, stopping when the second coordinate becomes 0. Prove that this process stops after finitely many steps.

Technique. Well-ordering: identify a quantity that strictly decreases with each step and cannot go below zero.

Discussion. Consider the sequence of second coordinates produced by the algorithm: starting from b, the next second coordinate is $a \bmod b$, which satisfies $0 \leq a \bmod b < b$. After one step the second coordinate has strictly decreased (or become 0 and stopped). At every subsequent step the same strict decrease happens.

Suppose, for a contradiction, the algorithm never stops. Then it produces an infinite strictly-decreasing sequence of non-negative integers $b > a \bmod b > \cdots$. The set of values in this sequence is a non-empty subset of $\mathbb{N}$, and by the well-ordering principle it has a least element b^*. But the value immediately after b^* in the sequence is strictly smaller than b^* and still non-negative, contradicting the minimality of b^*. So the algorithm must terminate. □

What to take away. Well-ordering says every non-empty subset of $\mathbb{N}$ has a least element. Its contrapositive says that $\mathbb{N}$ admits no infinite strictly decreasing sequence. Every termination proof for an iterative algorithm boils down to exhibiting a quantity that sits in $\mathbb{N}$ and strictly decreases at each step. For the Euclidean algorithm the quantity is the second coordinate; for the Tower of Hanoi in Problem 6.10, it is the count of discs still on the first peg.

STRUCTURAL INDUCTION

PROBLEM 6.10. In the Tower of Hanoi, n discs of distinct sizes are stacked on peg A in decreasing order from bottom to top. The goal is to transfer them all to peg C, using an auxiliary peg B, moving one disc at a time and never placing a larger disc on a smaller one. Prove that the minimum number of moves is exactly $2^n - 1$.

Technique. Two inductions in parallel: one showing $2^n - 1$ moves suffice, the other showing at least that many are required.

Discussion. Let $T(n)$ denote the minimum number of moves for n discs.

Upper bound: $T(n) \leq 2^n - 1$. Induction on n. Base $n = 1$: one move transfers the single disc from A to C, so $T(1) \leq 1 = 2^1 - 1$. For the step, suppose $T(n) \leq 2^n - 1$. With $n + 1$ discs the following procedure works: transfer the top n discs from A to B (at most $T(n)$ moves by induction, using C as auxiliary); move the bottom disc from A to C (one move); transfer the n discs from B to C (at most $T(n)$ moves by induction, using A as auxiliary). Total: at most $2T(n) + 1 \leq 2(2^n - 1) + 1 = 2^{n+1} - 1$.

Lower bound: $T(n) \geq 2^n - 1$. Induction on n. Base $n = 1$: at least one move is required to change the disc's location, so $T(1) \geq 1 = 2^1 - 1$. For the step, suppose $T(n) \geq 2^n - 1$. Consider any valid sequence of moves for $n + 1$ discs. The bottom disc must eventually move from A to C; at the instant it does, the other n discs must all be on B (nowhere else is legal). So the sequence splits into three phases: (i) move the top n discs from A to B, (ii) move the bottom disc from A to C, (iii) move the n discs from B to C. Phase (i) requires at least

$T(n)$ moves, phase (ii) requires 1, phase (iii) requires at least $T(n)$. Total: at least $2T(n) + 1 \geq 2(2^n - 1) + 1 = 2^{n+1} - 1$.

Combining the two bounds, $T(n) = 2^n - 1$. □

What to take away. The argument is really about the structure of any solution, not about any particular sequence of moves. The key observation is that at the moment the bottom disc travels from A to C, the remaining n discs must be stacked on the single auxiliary peg. This forces the problem to decompose into two n-disc subproblems and one large-disc move; everything else follows. Induction here is *structural*: it peels off the bottom disc, reduces to the n-disc case twice, and multiplies the counts. The same recursive peel-off pattern recurs whenever a combinatorial object can be decomposed into smaller copies of itself.

Chapter 7

Exponentials and logarithms

Computation through identities

Problem 7.1. Compute $\log_{10} \tan 1^\circ + \log_{10} \tan 2^\circ + \cdots + \log_{10} \tan 89^\circ$.

Technique. Pair complementary angles and exploit $\tan(90^\circ - x) = \cot x$.

Discussion. Pair the terms k and $90 - k$ for $k = 1, 2, \ldots, 44$, leaving the middle term $\log_{10} \tan 45^\circ$ unpaired. For each paired sum,

$$\log_{10} \tan k^\circ + \log_{10} \tan(90 - k)^\circ = \log_{10}(\tan k^\circ \cot k^\circ) = 0.$$

The middle term is $\log_{10} \tan 45^\circ = 0$ as well. Every contribution to the sum is therefore 0, and

$$\sum_{k=1}^{89} \log_{10} \tan k^\circ = 0.$$

□

What to take away. Long sums of logarithms collapse when the arguments multiply to a clean number. The complementary-angle identity $\tan(90^\circ - x) = \cot x$ is the symmetry here; in other problems it might be $\sin(90^\circ - x) = \cos x$, or some other pairing. The instruction is the same: scan the sum for terms that combine.

Problem 7.2. Compute $2^{\log_3 5} - 5^{\log_3 2}$.

Technique. Establish the identity $a^{\log_b c} = c^{\log_b a}$ for positive a, b, c with $b \neq 1$, and apply it.

Discussion. The identity follows by taking ln of each side:

$$\ln(a^{\log_b c}) \;=\; \log_b c \cdot \ln a \;=\; \frac{\ln a \cdot \ln c}{\ln b},$$

and the same expression results from $\ln(c^{\log_b a})$ by symmetry. Hence the two quantities are equal.

Setting $a = 2$, $b = 3$, $c = 5$ gives $2^{\log_3 5} = 5^{\log_3 2}$, so the difference is 0. □

What to take away. The identity $a^{\log_b c} = c^{\log_b a}$ is symmetric in a and c and dissolves any expression of the form "base raised to a logarithm." It is one of the two or three logarithmic identities the reader should be able to write down without thinking.

CHANGE OF BASE IN ALGEBRAIC DRESS

Problem 7.3. Let a, b be the lengths of the legs and c the length of the hypotenuse of a right triangle, with $c - b \neq 1$ and $c + b \neq 1$. Prove that

$$\log_{c+b} a + \log_{c-b} a \;=\; 2 \log_{c+b} a \cdot \log_{c-b} a.$$

Technique. Combine the change-of-base identity with the Pythagorean relation $a^2 = (c-b)(c+b)$.

Discussion. By change of base to natural logarithms,

$$\log_{c+b} a \;=\; \frac{\ln a}{\ln(c+b)}, \qquad \log_{c-b} a \;=\; \frac{\ln a}{\ln(c-b)}.$$

The left-hand side of the claim is therefore

$$\frac{\ln a}{\ln(c+b)} + \frac{\ln a}{\ln(c-b)} \;=\; \ln a \cdot \frac{\ln(c-b) + \ln(c+b)}{\ln(c+b)\,\ln(c-b)}.$$

The numerator simplifies via $\ln(c-b) + \ln(c+b) = \ln[(c-b)(c+b)]$. By Pythagoras, $(c-b)(c+b) = c^2 - b^2 = a^2$, so $\ln[(c-b)(c+b)] = \ln a^2 = 2 \ln a$. Substituting,

$$\log_{c+b} a + \log_{c-b} a \;=\; \frac{2(\ln a)^2}{\ln(c+b)\,\ln(c-b)}.$$

The right-hand side, by direct computation, equals

$$2\log_{c+b} a \cdot \log_{c-b} a \;=\; \frac{2(\ln a)^2}{\ln(c+b)\,\ln(c-b)},$$

the same expression. The two sides agree. □

What to take away. The identity is logarithmic, but the engine is Pythagoras: the geometric relation $a^2 = (c-b)(c+b)$, taken under ln, becomes a logarithmic relation that the change-of-base formula then unfolds. Whenever an algebraic identity in logarithms involves three quantities one of which is the product of the other two, look for a hidden multiplicative relationship in the underlying arguments.

Problem 7.4. Let $\alpha = \log_{12} 18$ and $\beta = \log_{24} 54$. Prove that $\alpha\beta + 5(\alpha-\beta) = 1$.

Technique. Express both logarithms in a common base via the prime factorisations of $12, 18, 24, 54$, then carry out polynomial algebra.

Discussion. Set $p = \ln 2$ and $q = \ln 3$. The four arguments factor as

$$12 = 2^2 \cdot 3, \quad 18 = 2 \cdot 3^2, \quad 24 = 2^3 \cdot 3, \quad 54 = 2 \cdot 3^3,$$

so $\alpha = (p+2q)/(2p+q)$ and $\beta = (p+3q)/(3p+q)$.

Compute $\alpha\beta$:

$$\alpha\beta \;=\; \frac{(p+2q)(p+3q)}{(2p+q)(3p+q)} \;=\; \frac{p^2+5pq+6q^2}{6p^2+5pq+q^2}.$$

Compute $\alpha-\beta$ over the common denominator $(2p+q)(3p+q)$:

$$\alpha-\beta \;=\; \frac{(p+2q)(3p+q)-(p+3q)(2p+q)}{(2p+q)(3p+q)}.$$

Expand the numerator. The first product is $3p^2+7pq+2q^2$; the second is $2p^2+7pq+3q^2$; their difference is p^2-q^2. Hence

$$\alpha-\beta \;=\; \frac{p^2-q^2}{6p^2+5pq+q^2}.$$

Adding $\alpha\beta$ and $5(\alpha-\beta)$ over the common denominator $6p^2+5pq+q^2$,

$$\alpha\beta+5(\alpha-\beta) \;=\; \frac{(p^2+5pq+6q^2)+5(p^2-q^2)}{6p^2+5pq+q^2}.$$

The numerator simplifies to $6p^2 + 5pq + q^2$, identical to the denominator, so $\alpha\beta + 5(\alpha - \beta) = 1$, as required. □

What to take away. A four-logarithm identity collapses to a polynomial identity in two variables once the prime factorisations are written down. The mechanical part (cross-multiplication and expansion) is unavoidable, but it always finishes; the algebraic miracle "the polynomial in p, q on the numerator equals the polynomial on the denominator" is what makes the original logarithmic claim true.

COMPARISON AND INEQUALITY

PROBLEM 7.5. Compare $\log_a 2$ and $\log_a 3$ for $a > 0$, $a \neq 1$.

Technique. Apply the change-of-base formula and case-split on the sign of $\ln a$.

Discussion. By change of base,

$$\log_a 2 \;=\; \frac{\ln 2}{\ln a}, \qquad \log_a 3 \;=\; \frac{\ln 3}{\ln a}.$$

Both numerators are positive constants with $\ln 2 < \ln 3$.

Case $a > 1$. Then $\ln a > 0$. Dividing $\ln 2 < \ln 3$ by the positive quantity $\ln a$ preserves the inequality, so $\log_a 2 < \log_a 3$.

Case $0 < a < 1$. Then $\ln a < 0$. Dividing $\ln 2 < \ln 3$ by a negative quantity reverses the inequality, so $\log_a 2 > \log_a 3$.

In summary: $\log_a 2 < \log_a 3$ for $a > 1$, and $\log_a 2 > \log_a 3$ for $0 < a < 1$. □

What to take away. The behaviour of $\log_a x$ flips at $a = 1$: for $a > 1$ the function is increasing, for $0 < a < 1$ it is decreasing. This is the same dichotomy that drives the sign-flip in the inequality at the end of the chapter; the source is the sign of $\ln a$, and case-splitting on it is non-negotiable whenever a is a parameter.

Problem 7.6. Prove that $|\log_b a + \log_a b| \geq 2$ for positive a, b with $a \neq 1$ and $b \neq 1$, with equality if and only if $a = b$ or $a = 1/b$.

Technique. Substitute $t = \log_b a$, reducing the inequality to a standard AM-GM bound on $t + 1/t$.

Discussion. Set $t = \log_b a$. The change-of-base identity gives $\log_a b = 1/\log_b a = 1/t$. The expression becomes

$$|\log_b a + \log_a b| \;=\; |t + \tfrac{1}{t}|.$$

Treat the two cases $t > 0$ and $t < 0$ separately.

Case $t > 0$. By AM-GM applied to t and $1/t$,

$$\frac{t + 1/t}{2} \;\geq\; \sqrt{t \cdot \tfrac{1}{t}} \;=\; 1, \qquad \text{so} \qquad t + 1/t \;\geq\; 2,$$

with equality iff $t = 1/t$, i.e. $t = 1$.

Case $t < 0$. Set $s = -t > 0$. Then $t + 1/t = -(s + 1/s) \leq -2$, again by AM-GM applied to s and $1/s$. So $|t + 1/t| \geq 2$, with equality iff $s = 1$, i.e. $t = -1$.

In both cases $|t + 1/t| \geq 2$, with equality iff $t = \pm 1$, i.e. iff $\log_b a = \pm 1$, i.e. iff $a = b$ or $a = 1/b$. $\square$

What to take away. The inequality $t + 1/t \geq 2$ for positive t is the two-variable arithmetic-geometric mean inequality: $(t + 1/t)/2 \geq \sqrt{t \cdot 1/t} = 1$, with equality only at $t = 1$. Logarithmic inequalities of this kind are almost always a disguised AM-GM application; the substitution $t = \log_b a$ is the key that exposes the mean-inequality form.

EQUATIONS

Problem 7.7. Solve $\log_2 x - 8 \log_{x^2} 2 \;=\; 3$ for $x > 0, x \neq 1$.

Technique. Substitute $u = \log_2 x$; rewrite $\log_{x^2} 2$ in terms of u via change of base; reduce to a quadratic in u.

Discussion. By change of base, $\log_{x^2} 2 = \frac{\ln 2}{\ln x^2} = \frac{\ln 2}{2 \ln x} = \frac{1}{2 \log_2 x}$. Setting $u = \log_2 x$ (note $u \neq 0$ because $x \neq 1$), the equation becomes

$$u - 8 \cdot \frac{1}{2u} = 3, \qquad \text{that is} \qquad u - \frac{4}{u} = 3.$$

Multiplying through by u,

$$u^2 - 3u - 4 = 0, \qquad (u-4)(u+1) = 0, \qquad u \in \{4, -1\}.$$

Translating back via $x = 2^u$: $u = 4$ gives $x = 16$, and $u = -1$ gives $x = 1/2$. Both lie in the domain $\{x > 0, x \neq 1\}$. Solutions: $x = 16$ or $x = 1/2$. $\square$

What to take away. The substitution $u = \log_a x$ converts most logarithmic equations into polynomial equations. Solve the polynomial, then exponentiate back. The only trap is the domain check; the polynomial may produce values of u that correspond to x outside the original domain. Here both work.

Problem 7.8. Solve $\dfrac{\log_8(8/x^2)}{(\log_8 x)^2} = 3$ for $x > 0, x \neq 1$.

Technique. Set $u = \log_8 x$; expand the numerator using $\log_8(8/x^2) = 1 - 2u$; reduce to a quadratic.

Discussion. The numerator simplifies as $\log_8(8/x^2) = \log_8 8 - \log_8 x^2 = 1 - 2u$. The equation becomes

$$\frac{1-2u}{u^2} = 3, \qquad \text{equivalently} \qquad 3u^2 + 2u - 1 = 0.$$

Factor: $(3u-1)(u+1) = 0$, so $u = 1/3$ or $u = -1$. Translating back via $x = 8^u$, the two values are $x = 2$ and $x = 1/8$. Both lie in the domain. Solutions: $x = 2$ or $x = 1/8$. $\square$

What to take away. The form was complicated, but every piece reduced to a function of $u = \log_8 x$. After substitution the algebra is mechanical. The domain restriction $x \neq 1$ corresponds to $u \neq 0$, which is needed to avoid division by zero in the original equation; both candidate u values are non-zero, so neither is rejected.

INEQUALITIES

PROBLEM 7.9. Solve $\log_{1/2} x + \log_3 x > 1$ for $x > 0, x \neq 1$.

Technique. Express both logarithms in a common base; the left-hand side becomes a constant times $\ln x$, and the sign of the constant determines whether the inequality flips on division.

Discussion. Using natural logarithms,

$$\log_{1/2} x = \frac{\ln x}{\ln(1/2)} = -\frac{\ln x}{\ln 2}, \qquad \log_3 x = \frac{\ln x}{\ln 3}.$$

The left-hand side becomes

$$\ln x\left(-\frac{1}{\ln 2} + \frac{1}{\ln 3}\right) = \ln x \cdot \frac{\ln 2 - \ln 3}{\ln 2 \cdot \ln 3}.$$

The factor $(\ln 2 - \ln 3)/(\ln 2 \cdot \ln 3)$ is *negative*, because $\ln 2 < \ln 3$ while both denominator factors are positive. Call this factor $-k$ with $k > 0$. The inequality becomes $-k \ln x > 1$, equivalently

$$\ln x < -\frac{1}{k} = \frac{\ln 2 \cdot \ln 3}{\ln 2 - \ln 3}.$$

The right-hand side is negative (numerator positive, denominator negative), so this forces $\ln x < 0$, that is $x < 1$. Combined with $x > 0$,

$$0 < x < \exp\left(\frac{\ln 2 \cdot \ln 3}{\ln 2 - \ln 3}\right),$$

numerically $0 < x < 0.152\ldots$. □

What to take away. The step that traps almost every student is the sign of the coefficient on $\ln x$. Because $\log_{1/2}$ is decreasing, dividing through by its coefficient flips the inequality. Whenever a logarithm with base less than 1 appears on the left, expect a sign flip, and verify that the final domain restriction $x > 0$ is consistent with the resulting bound on $\ln x$.

PROBLEM 7.10. Solve $x^{1/\log_{10} x} \cdot \log_{10} x < 1$ for $x > 0, x \neq 1$.

Technique. Recognise the identity $x^{1/\log_{10} x} = 10$ and reduce to a one-variable linear inequality in $\log_{10} x$.

Discussion. Take $\log_{10}$ of $x^{1/\log_{10} x}$:

$$\log_{10}(x^{1/\log_{10} x}) \;=\; \frac{1}{\log_{10} x} \cdot \log_{10} x \;=\; 1,$$

so $x^{1/\log_{10} x} \;=\; 10^1 \;=\; 10$ for every valid x. The inequality therefore reads $10 \log_{10} x < 1$, that is

$$\log_{10} x \;<\; \tfrac{1}{10}, \qquad \text{equivalently} \qquad x \;<\; 10^{1/10}.$$

Combined with the domain $x > 0$, $x \neq 1$,

$$x \in (0, 1) \cup (1, 10^{1/10}).$$

□

What to take away. The identity $x^{1/\log_a x} = a$ is one of the cleanest collapses in the logarithmic toolkit. Faced with an expression of this shape, always check whether the exponent and the base conspire to a constant; what looks transcendental often is not. The domain check at the end is non-negotiable: the original expression is undefined at $x = 1$, so the point must be excluded from the solution.

Part II

Algebra

CHAPTER 8

Quadratic equations and inequalities

PARAMETER CONDITIONS FOR QUADRATIC INEQUALITIES

PROBLEM 8.1. Find all real values of m for which the inequality

$$mx^2 - 4x + 3m + 1 \; > \; 0$$

holds for every $x > 0$.

Discussion. The first instinct most students have is to "apply the discriminant condition." But this would be wrong. The discriminant test is for a quadratic to be positive on *all* of $\mathbb{R}$. Here we are only asked about the positive ray $x > 0$. A quadratic can dip below zero on $x < 0$ and still be positive on $x > 0$. So we must look at the actual picture of the parabola, not just memorise a formula.

Split into cases by the sign of m, because m decides whether we even have a quadratic, and if so, which way the parabola opens.

Case $m = 0$. The expression becomes $-4x + 1$, a line with slope -4. It is positive only for $x < \frac{1}{4}$, so it fails at (say) $x = 1$. So $m = 0$ does not work.

Case $m < 0$. The leading coefficient is negative, so the parabola opens downward. As $x \to +\infty$, the mx^2 term dominates and drags the expression to $-\infty$. The inequality fails for sufficiently large x. Reject.

Case $m > 0$**.** The parabola opens upward. Two things might happen.

Sub-case 1: the parabola never touches the x-axis. Then $mx^2 - 4x + 3m + 1 > 0$ for *every* real x, and in particular for every $x > 0$. The condition is: discriminant negative.

Compute:

$$D = 16 - 4m(3m + 1) = -4(3m + 4)(m - 1).$$

Since $m > 0$, the factor $(3m + 4)$ is positive. So $D < 0$ requires $(m - 1) > 0$, i.e., $m > 1$. Thus for every $m > 1$ the inequality holds.

Sub-case 2: the parabola has two real roots. This occurs when $D \geq 0$, i.e., when $0 < m \leq 1$. Call the roots x_1 and x_2. The parabola (opening upward) is negative strictly between them. For the inequality to still hold on $(0, \infty)$, both roots would need to be on the negative ray (so the dip happens only for $x < 0$).

Use Viète's formulas: for a quadratic $ax^2 + bx + c$ with roots x_1, x_2, the sum is $-b/a$ and the product is c/a. For $mx^2 - 4x + 3m + 1 = 0$,

$$x_1 + x_2 = \frac{4}{m} > 0, \qquad x_1 x_2 = \frac{3m + 1}{m} = 3 + \frac{1}{m} > 0.$$

Sum and product are both positive, so both roots are positive. The parabola dips below zero between two points on $(0, \infty)$, and the inequality fails there. Reject this sub-case.

Conclusion.

$$m > 1.$$

The key step was not the discriminant calculation; it was recognising that "positive on $\mathbb{R}$" and "positive on $(0, \infty)$" are genuinely different conditions, and that we need Viète's formulas to locate the roots without solving for them.

A related warning. A negative discriminant means a quadratic does not cross the x-axis, but it does not tell you on which side the parabola sits. The side is decided by the leading coefficient. For instance $-x^2 - 2x - 5$ has discriminant $-16 < 0$, but the parabola opens downward and sits entirely *below* the axis, not above.

Problem 8.2. Find all real values of m for which the quadratic $f(x) = x^2 + mx + m^2 + 6m$ is negative for every x with $1 < x < 2$.

Discussion. The leading coefficient is $1 > 0$, so the parabola opens upward. An upward-opening parabola is negative strictly between its two real roots, and only there. So the question becomes: for which m does f have two real roots $r_1 < r_2$ with the interval $(1,2)$ nested inside (r_1, r_2)?

Here is a clean geometric criterion. An upward-opening parabola is ≤ 0 at a point exactly when that point lies between (or at) its two roots. Hence $(1,2) \subseteq [r_1, r_2]$ if and only if

$$f(1) \leq 0 \qquad \text{and} \qquad f(2) \leq 0.$$

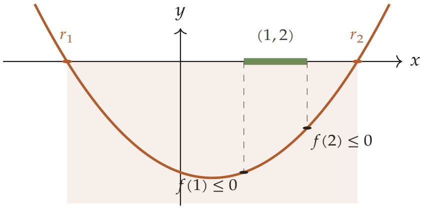

Figure 8.1: The parabola $y = f(x)$ (drawn for the concrete case $m = -1$, giving $f(x) = x^2 - x - 5$). The two roots are r_1, r_2 (copper); the interval $(1,2)$ (sage) must sit inside (r_1, r_2). Equivalently, $f(1)$ and $f(2)$ (both dots below the axis) must both be ≤ 0.

Why "≤ 0" rather than "< 0" at the endpoints? If $f(1) = 0$ then $x = 1$ is a root of f, but the parabola drops strictly below zero for every x just to the right of 1, so f is strictly negative on the whole open interval $(1,2)$. Using strict inequality at the endpoints would cut off the boundary values of m and shrink the answer.

Why this one criterion is enough. You might worry about additional conditions: the discriminant must be positive (roots exist); the vertex's x-coordinate $-b/(2a) = -m/2$ must lie in $(1,2)$; and so on. In fact $f(1) \leq 0$ and $f(2) \leq 0$ settle everything by themselves, and the reason is *convexity*, not the position of the vertex. An upward parabola is a convex function, so on $[1,2]$ its graph lies on or below the chord joining $(1,f(1))$ and $(2,f(2))$. Both endpoints lie at or below the axis, so that chord lies at or below the axis, and therefore $f(x) \leq 0$ for every $x \in [1,2]$. The whole interval is thus trapped between the two roots, which in particular must exist. Note the vertex need *not* lie in $[1,2]$ for this to happen, and sometimes it does not. Two conditions suffice.

Solving the two inequalities. Compute:

$$f(1) = 1 + m + m^2 + 6m \;=\; m^2 + 7m + 1,$$
$$f(2) = 4 + 2m + m^2 + 6m \;=\; m^2 + 8m + 4.$$

Each is a quadratic in m. Rearrange each as a completed square in m.

$$m^2 + 7m + 1 \;=\; (m + \tfrac{7}{2})^2 - \tfrac{45}{4}.$$

So $m^2 + 7m + 1 \leq 0$ when $(m + \frac{7}{2})^2 \leq \frac{45}{4}$, i.e.,

$$\frac{-7 - 3\sqrt{5}}{2} \;\leq\; m \;\leq\; \frac{-7 + 3\sqrt{5}}{2}.$$

Similarly,

$$m^2 + 8m + 4 \;=\; (m + 4)^2 - 12,$$

so $m^2 + 8m + 4 \leq 0$ when $(m + 4)^2 \leq 12$, i.e.,

$$-4 - 2\sqrt{3} \;\leq\; m \;\leq\; -4 + 2\sqrt{3}.$$

Numerically the first interval is about $[-6.854, -0.146]$ and the second about $[-7.464, -0.536]$. Their intersection is the narrower interval

$$\frac{-7 - 3\sqrt{5}}{2} \;\leq\; m \;\leq\; -4 + 2\sqrt{3}.$$

The lesson: before grinding out a discriminant or quadratic formula, ask what the geometry of the parabola tells you. Here it told us that only two conditions, $f(1) \leq 0$ and $f(2) \leq 0$, fully characterise the parameter m. Any additional condition one might worry about (existence of real roots, vertex location, leading coefficient sign) is either already implied or does not apply.

ROOT LOCATION

PROBLEM 8.3. Find all real values of d for which both roots of

$$x^2 - 6dx + 2 - 2d + 9d^2 \ = \ 0$$

are real and strictly greater than 3.

Discussion. Parabola opens upward. For a parabola opening upward, a test point k lies outside the interval between the roots iff $f(k) > 0$. For both roots to exceed 3, three conditions must hold:

(i) Discriminant ≥ 0 (real roots exist);

(ii) $f(3) > 0$ (so 3 is outside the interval between the roots, specifically to the left of the smaller);

(iii) The vertex's x-coordinate, $-b/(2a) = 3d$, is > 3 (the parabola is shifted right far enough).

Discriminant:

$$D \ = \ (6d)^2 - 4(2-2d+9d^2) \ = \ 36d^2 - 8 + 8d - 36d^2 \ = \ 8d - 8.$$

So $D \geq 0$ iff $d \geq 1$.

$f(3) = 9-18d+2-2d+9d^2 = 9d^2-20d+11 = (9d-11)(d-1)$. So $f(3) > 0$ iff $d < 1$ or $d > 11/9$.

Vertex: $3d > 3$ iff $d > 1$.

Combining all three: $d \geq 1$ and $d > 1$ and ($d < 1$ or $d > 11/9$) give $d > 11/9$. (At $d = 11/9$, $f(3) = 0$, so 3 is a root; it does not strictly exceed 3.)

$$d > \tfrac{11}{9}.$$

What to take away. "Both roots greater than k" for an upward parabola needs three conditions: real roots ($D \geq 0$), the test value lies outside the root interval ($f(k) > 0$), and the vertex lies to the right of k. Dropping any one of these lets in spurious cases. For instance, $f(k) > 0$ on its own is consistent with both roots lying *below* k rather than above.

Problem 8.4. For which real values of a does the equation

$$(a^2 + a + 1)x^2 + (a - 1)x + a^2 = 0$$

have one root greater than 3 and the other less than 3?

Discussion. Leading coefficient $a^2+a+1 = (a+1/2)^2+3/4 > 0$ always, so the parabola opens upward. "3 strictly between the two roots" for an upward parabola means $f(3) < 0$.

$$f(3) = 9(a^2 + a + 1) + 3(a - 1) + a^2 = 10a^2 + 12a + 6.$$

We need this < 0. Discriminant of $10a^2 + 12a + 6$ is $144 - 240 = -96 < 0$, and leading coefficient is positive; so $10a^2 + 12a + 6 > 0$ for every real a. It is never negative.

$$\text{no real } a \text{ works.}$$

What to take away. A quadratic inequality $p(a) < 0$ is impossible if p has no real roots and positive leading coefficient. Some problems have no valid parameter values, and that is a complete answer, not an indication of a mistake.

Problem 8.5. For which real values of a do the roots x_1, x_2 of

$$2x^2 - 2(2a + 1)x + a(a - 1) = 0$$

satisfy $x_1 < a < x_2$?

Discussion. Same criterion, with the test point now a parameter: a strictly between the roots iff $f(a) < 0$ (leading coefficient $2 > 0$). Compute:

$$f(a) = 2a^2 - 2a(2a + 1) + a(a - 1) = -a^2 - 3a = -a(a + 3).$$

We need $-a(a+3) < 0$, i.e., $a(a+3) > 0$, which holds for $a > 0$ or $a < -3$.

$$a < -3 \text{ or } a > 0.$$

What to take away. The "test-point between the roots" criterion works whether the test point is a fixed number or a parameter. The algebra is the same.

REDUCING TO A POLYNOMIAL BY SUBSTITUTION

PROBLEM 8.6. Solve $4^x = 2 \cdot 14^x + 3 \cdot 49^x$.

Discussion. A quadratic hides inside this. The three bases $4, 14, 49$ factor as $2^2, 2 \cdot 7, 7^2$. Divide through by 49^x:

$$\left(\frac{4}{49}\right)^x = 2\left(\frac{14}{49}\right)^x + 3, \qquad \text{i.e.,} \qquad \left(\frac{2}{7}\right)^{2x} = 2\left(\frac{2}{7}\right)^x + 3.$$

Let $y = (2/7)^x$; the equation becomes $y^2 = 2y + 3$, or $y^2 - 2y - 3 = 0$, which factors as $(y-3)(y+1) = 0$. So $y = 3$ or $y = -1$. Only $y = 3$ is a valid value of $(2/7)^x$ (since $(2/7)^x > 0$).

Solve: $(2/7)^x = 3$, so

$$x = \log_{2/7} 3 = \frac{\log 3}{\log 2 - \log 7}.$$

What to take away. When three exponentials have bases in geometric progression (here $4, 14, 49$ with common ratio $7/2$), divide through by the largest-base term and substitute. The equation becomes a quadratic in the ratio, and what looked transcendental collapses to material from this chapter.

PROBLEM 8.7. Show that $8^x - 3 \cdot 4^x - 3 \cdot 2^x + 8 = 0$ has exactly two real solutions, and locate them.

Discussion. Set $y = 2^x$. Then $4^x = y^2$ and $8^x = y^3$, and the equation becomes

$$y^3 - 3y^2 - 3y + 8 = 0,$$

a cubic in y. By the rational-root theorem, any rational root p/q (in lowest terms) must have $p \mid 8$ and $q \mid 1$, so $y \in \{\pm 1, \pm 2, \pm 4, \pm 8\}$. None of these yield zero (e.g., $y = 1$ gives $1 - 3 - 3 + 8 = 3$; $y = 2$ gives -2). So the cubic has no rational root.

Count its real roots by sign changes. Evaluate the cubic at $y = 0, 3, 4$: the values are $8, -1, 12$. The sign changes from $+$ to $-$ between 0 and 3, and from $-$ to $+$ between 3 and 4, so there is a root in $(0,3)$ and a root in $(3,4)$. As $y \to -\infty$ the cubic tends to $-\infty$ while its value at 0 is $8 > 0$, so there is a third root in $(-\infty, 0)$. That is three real roots, and a cubic has no more, so these are all of them: two positive and one negative. (Consistently, the three roots multiply to -8, the constant term negated, which a pair of positive roots and one negative root indeed do.) Numerically the two positive roots are about 1.52 and 3.15.

Since $y = 2^x > 0$, only the two positive roots count, and each gives exactly one solution $x = \log_2 y$. So the equation has exactly two real solutions, $x = \log_2 y$ for $y \approx 1.52$ and $y \approx 3.15$, that is $x \approx 0.61$ and $x \approx 1.65$. (Exact closed forms exist through Cardano's formula but are unilluminating; the point here is the count, two, established without them.)

What to take away. After substitution converts an exponential equation into a polynomial, check rational-root candidates first (they come from a single list). If none work, sign-change arguments count the real roots without naming them. In easier problems the rational-root check will usually succeed; in harder ones it may not, and the sign-change argument is what saves you.

Viète's formulas and symmetric functions

Problem 8.8. Let a, b be the roots of $x^2 + px + 1 = 0$ and let c, d be the roots of $x^2 + qx + 1 = 0$. Show that $q^2 - p^2 = (a-c)(b-c)(a+d)(b+d)$.

Discussion. Recall Viète's formulas for a monic polynomial: for $x^n + a_1 x^{n-1} + \cdots + a_n$ with roots $\alpha_1, \ldots, \alpha_n$,

$$\sum_i \alpha_i = -a_1, \quad \sum_{i<j} \alpha_i \alpha_j = a_2, \quad \ldots, \quad \prod_i \alpha_i = (-1)^n a_n.$$

Any symmetric function of the roots can thus be read off from the coefficients, without solving for the roots themselves.

For this problem, don't solve either quadratic explicitly; that leads to messy radicals. The right-hand side factorises into two pair-symmetric products, which Viète's handles cleanly:

$$\begin{aligned}(a-c)(b-c) &= ab - c(a+b) + c^2 \;=\; 1 + pc + c^2,\\ (a+d)(b+d) &= ab + d(a+b) + d^2 \;=\; 1 - pd + d^2.\end{aligned}$$

Now c, d are roots of $x^2 + qx + 1 = 0$, so $c^2 = -qc - 1$ and $d^2 = -qd - 1$. Substitute:

$$\begin{aligned}1 + pc + c^2 &= 1 + pc - qc - 1 = (p-q)c,\\ 1 - pd + d^2 &= 1 - pd - qd - 1 = -(p+q)d.\end{aligned}$$

Multiplying, the right-hand side is $(p-q)c \cdot [-(p+q)d] = -(p^2 - q^2)cd$. Viète's on c, d gives $cd = 1$, so the right-hand side equals $q^2 - p^2$. Done.

What to take away. When an expression involves roots of two quadratics, regroup it into pair-symmetric factors before simplifying. Viète's then turns quadratic substitutions into linear ones.

Problem 8.9. Let α, β be the roots of $ax^2 + bx + c = 0$ and $\alpha + \delta, \beta + \delta$ the roots of $Ax^2 + Bx + C = 0$. Show that $(b^2 - 4ac)/a^2 = (B^2 - 4AC)/A^2$.

Discussion. Shifting both roots by the same constant doesn't change the gap between them: $(\alpha + \delta) - (\beta + \delta) = \alpha - \beta$. So the two sides of the claimed identity, if they equal anything, should both equal $(\alpha - \beta)^2$.

For a general quadratic $ax^2 + bx + c$ with roots α, β,

$$(\alpha-\beta)^2 \;=\; (\alpha+\beta)^2 - 4\alpha\beta \;=\; \left(-\frac{b}{a}\right)^2 - 4\cdot\frac{c}{a} \;=\; \frac{b^2-4ac}{a^2}.$$

Applying this to both quadratics, the two sides of the identity *both* equal $(\alpha - \beta)^2$. Done.

What to take away. $(\alpha - \beta)^2 = (b^2 - 4ac)/a^2$ is worth memorising. It turns any root-gap question into a coefficient calculation.

Problem 8.10. Find the sum of the cubes of the roots of $x^4 - 2x^3 + x^2 - 5x + 1 = 0$.

Discussion. No quartic formula to apply. Call the roots $\alpha, \beta, \gamma, \delta$, and denote their elementary symmetric functions σ_k and power sums s_k. Reading σ_k off the coefficients,

$$\sigma_1 = 2, \qquad \sigma_2 = 1, \qquad \sigma_3 = 5, \qquad \sigma_4 = 1.$$

(The signs come from $\sigma_k = (-1)^k a_k$ where a_k is the coefficient of x^{n-k}.)

Newton's identities express s_k inductively in terms of the σ_j's. For the first three,

$$\begin{aligned} s_1 &= \sigma_1, \\ s_2 &= \sigma_1 s_1 - 2\sigma_2, \\ s_3 &= \sigma_1 s_2 - \sigma_2 s_1 + 3\sigma_3. \end{aligned}$$

Compute: $s_1 = 2$, then $s_2 = (2)(2) - 2(1) = 2$, then

$$s_3 \;=\; (2)(2) - (1)(2) + 3(5) \;=\; 4 - 2 + 15 \;=\; 17.$$

What to take away. Newton's identities bridge the elementary symmetric functions (which you read off from the coefficients) and the power-sum symmetric functions (which problems usually ask about). Once you internalise the pattern, sum-of-powers questions on polynomial roots take three lines.

CHAPTER 9

Polynomial equations and factor theorem

REMAINDER THEOREM

PROBLEM 9.1. A polynomial in x of degree greater than three leaves remainders 2, 1 and -1 when divided respectively by $(x-1)$, $(x+2)$ and $(x+1)$. What remainder does it leave when divided by $(x-1)(x+2)(x+1)$?

Discussion. The remainder theorem states: when a polynomial $p(x)$ is divided by $(x-\alpha)$, the remainder equals $p(\alpha)$. So the three given facts become

$$p(1)=2, \qquad p(-2)=1, \qquad p(-1)=-1.$$

When $p(x)$ is divided by the cubic $(x-1)(x+2)(x+1)$, the remainder $r(x)$ has degree at most 2, say

$$r(x) \;=\; ax^2+bx+c.$$

Write $p(x) = q(x)(x-1)(x+2)(x+1)+r(x)$. Substituting $x = 1, -2, -1$ into this identity kills the $q(x)$ term (because $(x-1)(x+2)(x+1)=0$ at each of these points), leaving

$$p(1)=r(1), \qquad p(-2)=r(-2), \qquad p(-1)=r(-1).$$

These are three linear equations in a, b, c:

$$a+b+c=2,$$

$$\begin{aligned} 4a - 2b + c &= 1, \\ a - b + c &= -1. \end{aligned}$$

Subtract the third from the first: $2b = 3$, so $b = 3/2$. Subtract the second from the first: $-3a + 3b = 1$, so $a = b - 1/3 = 3/2 - 1/3 = 7/6$. Finally, $c = 2 - a - b = 2 - 7/6 - 3/2 = -2/3$. Hence

$$r(x) \;=\; \tfrac{7}{6}x^2 + \tfrac{3}{2}x - \tfrac{2}{3}.$$

What to take away. The remainder on division by a product of linear factors has degree at most one less than the divisor. Apply the remainder theorem at each zero of the divisor to recover the unknown coefficients of $r(x)$ linearly. No long division, no synthetic division, no cleverness, just substitutions and a linear system.

Problem 9.2. Find the remainder when x^{100} is divided by $x^2 - 3x + 2$.

Discussion. The divisor factors as $(x - 1)(x - 2)$, with roots 1 and 2. The remainder $r(x)$ has degree at most 1, so write $r(x) = Ax + B$. From $x^{100} = q(x)(x - 1)(x - 2) + Ax + B$, substituting the two roots gives

$$1^{100} \;=\; A + B, \qquad 2^{100} \;=\; 2A + B.$$

Subtracting, $A = 2^{100} - 1$, and $B = 1 - A = 2 - 2^{100}$. So

$$r(x) \;=\; (2^{100} - 1)x + (2 - 2^{100}) \;=\; (2^{100} - 1)(x - 1) + 1.$$

What to take away. Long division of x^{100} by a quadratic is impossible in practice. The remainder theorem at each root of the divisor gives a linear system for the coefficients of $r(x)$, and this works no matter how large the degree of the dividend.

FACTOR THEOREM

PROBLEM 9.3. Suppose $x^2 + px + 1$ is a factor of $ax^3 + bx + c$, where $a \neq 0$. Show that $a^2 - c^2 = ab$.

Discussion. The factor theorem states: $(x - \alpha)$ is a factor of a polynomial $P(x)$ if and only if $P(\alpha) = 0$. This is just the remainder theorem in the case when the remainder is zero. It generalises from linear factors to polynomial factors: if $g(x)$ is a factor of $P(x)$ then every root of $g(x) = 0$ is a root of $P(x) = 0$.

For this problem, the cleanest route uses polynomial division with undetermined coefficients. If $x^2 + px + 1$ is a factor of $ax^3 + bx + c$, then

$$ax^3 + bx + c \;=\; (x^2 + px + 1)(Ax + B)$$

for some constants A, B. Expand the right-hand side:

$$(x^2 + px + 1)(Ax + B) \;=\; Ax^3 + (B + Ap)x^2 + (A + Bp)x + B.$$

Match coefficients with $ax^3 + 0 \cdot x^2 + bx + c$:

$$\begin{aligned} A &= a && \text{(coefficient of } x^3\text{)} \\ B + Ap &= 0 && \text{(coefficient of } x^2\text{)} \\ A + Bp &= b && \text{(coefficient of } x^1\text{)} \\ B &= c && \text{(constant term)} \end{aligned}$$

From the first and last, $A = a, B = c$. The second says $c + ap = 0$, so $p = -c/a$. The third then reads $a + cp = b$, i.e., $a - c^2/a = b$. Multiplying both sides by a, which is nonzero by hypothesis,

$$a^2 - c^2 = ab.$$

What to take away. When asked whether a polynomial is a factor of a higher-degree polynomial, set up the quotient with undetermined coefficients and match powers. This is cleaner than polynomial long division because the steps are all linear. The factor theorem through its root interpretation is the deeper statement, but for computation the matching-coefficients method is usually faster.

Problem 9.4. Show that $x^2 + x + 1$ divides

$$x^{3m} + x^{3n+1} + x^{3p+2}$$

for every choice of non-negative integers m, n, p.

Discussion. Divisibility by a quadratic cannot be checked by a single real substitution. But the factor theorem generalises: a monic polynomial $g(x)$ divides $P(x)$ iff every root of g (real or complex) is a root of P. We can avoid invoking complex numbers here and work purely with polynomial remainders.

Modulo x^2+x+1, the divisor gives the identity $x^2 \equiv -x-1$, and hence

$$x^3 \;=\; x \cdot x^2 \;\equiv\; x(-x-1) \;=\; -x^2 - x \;\equiv\; -(-x-1) - x \;=\; 1.$$

So $x^3 \equiv 1 \pmod{x^2+x+1}$. Reducing each term,

$$(x^3)^m + (x^3)^n x + (x^3)^p x^2 \;\equiv\; 1 + x + x^2 \;\equiv\; 0.$$

The remainder on dividing by x^2+x+1 is zero, which is exactly what it means for x^2+x+1 to be a factor. □

What to take away. When the divisor $g(x)$ is irreducible over $\mathbb{R}$ (here $x^2 + x + 1$ has no real roots), you can still apply the factor theorem *algebraically*, by working modulo $g(x)$. The identity $x^3 \equiv 1 \pmod{x^2+x+1}$ is worth remembering; it collapses any polynomial in x into a residue of degree at most one.

REPEATED ROOTS

Problem 9.5. Find all real values of a for which the polynomial $P(x) = x^3 - 3x + a$ has a repeated real root.

Discussion. A repeated root r means $(x-r)^2$ divides $P(x)$. By the factor theorem applied to the quotient, this is equivalent to

$$P(r) = 0 \qquad \text{and} \qquad P'(r) = 0.$$

(If $(x-r)^2$ divides $P(x)$, write $P(x) = (x-r)^2 Q(x)$; differentiate to get $P'(x) = 2(x-r)Q(x) + (x-r)^2 Q'(x)$, so $P'(r) = 0$ as well. The converse, that $P(r) = P'(r) = 0$ implies $(x-r)^2 \mid P(x)$, follows from the factor theorem applied to P' and then to P.)

Compute $P'(x) = 3x^2 - 3 = 3(x-1)(x+1)$. So any repeated root r must satisfy $r = 1$ or $r = -1$. Substitute into P:

$$P(1) = 1 - 3 + a = a - 2, \qquad P(-1) = -1 + 3 + a = a + 2.$$

So P has a repeated root at $r = 1$ iff $a = 2$, and a repeated root at $r = -1$ iff $a = -2$.

$$a = 2 \text{ or } a = -2.$$

(Verification: $x^3 - 3x + 2 = (x-1)^2(x+2)$ and $x^3 - 3x - 2 = (x+1)^2(x-2)$.)

What to take away. "Repeated root" is equivalent to "root of both P and P'." This reduces double-root problems to two simultaneous factor-theorem conditions, avoiding any discriminant-of-cubic machinery. The same idea extends: a root of multiplicity k is a root of $P, P', P'', \dots, P^{(k-1)}$.

Problem 9.6. Let $n \geq 2$ be a positive integer. Find all real a and b for which $(x-1)^2$ divides $x^n + ax + b$.

Discussion. Same criterion as the previous problem, but now the coefficients are the unknowns and the repeated root is fixed. Let $P(x) = x^n + ax + b$. Requiring $(x-1)^2 \mid P(x)$ is equivalent to $P(1) = 0$ and $P'(1) = 0$.

$$P(1) \;=\; 1 + a + b \;=\; 0, \qquad P'(1) \;=\; n + a \;=\; 0.$$

The second gives $a = -n$; substituting into the first gives $b = -1 - a = n - 1$.

$$a = -n, \qquad b = n - 1.$$

(Verification: $x^n - nx + (n-1) = (x-1)[x^{n-1} + x^{n-2} + \cdots + x + 1 - n \cdot \frac{x-1}{x-1}]$ ...rather than spell out the quotient, one can observe that $x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$ and $n(x-1)$ is divisible by $x - 1$, so $x^n - nx + (n-1) = (x^n - 1) - n(x-1)$ is divisible by $x-1$; dividing by $x-1$ gives $x^{n-1}+x^{n-2}+\cdots+x+1-n$, which evaluates to 0 at $x=1$, confirming the second factor of $(x-1)$.)

What to take away. The repeated-root criterion runs in both directions: given the polynomial, find the root (previous problem); given the root, find the coefficients (this one). Both reduce to the same two equations, $P(r)=0$ and $P'(r)=0$.

TRANSFORMING ROOTS

PROBLEM 9.7. Let α, β, γ be the roots of $x^3+px+q=0$ with $q \neq 0$. Find a polynomial whose roots are $1/\alpha, 1/\beta, 1/\gamma$.

Discussion. If $y=1/x$ then $x=1/y$. The new polynomial should vanish at $y=1/\alpha$, that is, at values of y for which $1/y=\alpha$ is a root of the original. Substitute $x=1/y$ into the original equation:

$$\left(\frac{1}{y}\right)^3+p\cdot\frac{1}{y}+q \;=\; 0.$$

Multiply through by y^3 (permissible since $y=1/\alpha$ is non-zero, because $q\neq 0$ means $\alpha\neq 0$):

$$qy^3+py^2+1 \;=\; 0.$$

What to take away. The reciprocal transform $x \mapsto 1/x$ combined with clearing denominators reverses the coefficient string of a polynomial: the kth and $(n-k)$th coefficients swap. If the original polynomial is *palindromic* (coefficients the same read forwards and backwards), it is its own reciprocal, so its roots come in reciprocal pairs $\alpha, 1/\alpha$. That observation reduces many apparent quartics to quadratics.

PROBLEM 9.8. Let α, β, γ be the roots of $x^3+px+q=0$. Find a polynomial whose roots are $\alpha^2, \beta^2, \gamma^2$.

Discussion. Use Viète's formulas on the original: $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\gamma\alpha=p$, $\alpha\beta\gamma=-q$. Now compute the elementary symmetric functions of $\alpha^2, \beta^2, \gamma^2$. For the first, square the sum and subtract twice the pair-sum:

$$\sum \alpha^2 \;=\; \left(\sum \alpha\right)^2 - 2\sum \alpha\beta \;=\; 0 - 2p \;=\; -2p.$$

For the second, square the pair-sum and subtract twice the product-times-sum:

$$\sum \alpha^2\beta^2 \;=\; \left(\sum \alpha\beta\right)^2 - 2\alpha\beta\gamma \sum \alpha \;=\; p^2.$$

For the third, $\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2 = q^2$. A monic cubic with roots $\alpha^2, \beta^2, \gamma^2$ therefore has coefficients $-(-2p), +p^2, -(q^2)$ (signs from Viète's):

$$y^3 + 2py^2 + p^2y - q^2 \;=\; 0.$$

What to take away. To transform roots by an algebraic rule $\alpha \mapsto f(\alpha)$, compute the elementary symmetric functions of the new roots in terms of those of the old, and read the new polynomial off Viète's. For squaring, the key identities are $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$ and $\sum \alpha^2\beta^2 = (\sum \alpha\beta)^2 - 2\alpha\beta\gamma \sum \alpha$; for cubing, Newton's identities (previous chapter).

Polynomials with integer coefficients

Problem 9.9. Let $P(x)$ be a polynomial with integer coefficients and let a, b be distinct integers. Show that $(a - b)$ divides $P(a) - P(b)$.

Discussion. Write $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ with $c_k \in \mathbb{Z}$. Then

$$P(a) - P(b) \;=\; \sum_{k=0}^{n} c_k(a^k - b^k).$$

The factor theorem (applied with y treated as a constant) tells us that $(a - b) \mid (a^k - b^k)$ for every $k \geq 1$: explicitly,

$$a^k - b^k \;=\; (a - b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1}).$$

The bracketed factor is an integer. So $(a-b)$ divides every term of the sum, and hence divides the sum itself. □

What to take away. This is the number-theoretic face of the factor theorem. It turns polynomial values at integer inputs into a divisibility constraint: $P(a) \equiv P(b) \pmod{a-b}$ for any integer polynomial P and any integers a, b. The next problem is the canonical consequence.

Problem 9.10. Suppose $P(x)$ is a polynomial with integer coefficients such that $P(a) = P(b) = P(c) = 1$ for three distinct integers a, b, c. Prove that $P(x) = 0$ has no integer solution.

Discussion. Suppose, for contradiction, $P(r) = 0$ for some integer r. Apply the previous problem with $(a, r), (b, r), (c, r)$:

$$(a-r) \mid P(a) - P(r) = 1, \qquad (b-r) \mid 1, \qquad (c-r) \mid 1.$$

So each of $a-r, b-r, c-r$ is a divisor of 1 in $\mathbb{Z}$, hence ± 1. Three distinct integers a, b, c give three distinct values $a-r, b-r, c-r$, but $\{\pm 1\}$ contains only two integers. Contradiction.

Hence no such integer r exists. □

What to take away. Integer-polynomial problems are usually solved by this *pigeonhole on divisors* strategy. A polynomial with integer coefficients that takes small values (like ± 1) at too many integer points is sharply constrained, often so sharply that it cannot take the value 0 at any integer. The same argument, with small changes, shows that a polynomial with integer coefficients cannot take the value 0 and the value 2 at integer inputs differing by anything other than 1 or 2.

Chapter 10

Complex numbers

Cartesian and polar forms

Problem 10.1. Find all complex numbers z satisfying

$$\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}, \qquad \left|\frac{z-4}{z-8}\right| = 1.$$

Technique. Write $z = x + iy$; each modulus equation becomes a real equation in x and y; intersect the two loci.

Discussion. The second equation, $|z-4| = |z-8|$, says that z is equidistant from the real points 4 and 8. This is the perpendicular bisector, the vertical line $x = 6$.

The first equation, $|z-12| = \frac{5}{3}|z-8i|$, squares to

$$9[(x-12)^2 + y^2] = 25[x^2 + (y-8)^2].$$

Expanding and simplifying,

$$16x^2 + 16y^2 + 216x - 400y + 304 = 0,$$

equivalently $x^2 + y^2 + \frac{27}{2}x - 25y + 19 = 0$. Substituting $x = 6$,

$$36 + y^2 + 81 - 25y + 19 = 0, \qquad y^2 - 25y + 136 = 0,$$

with roots $y = \frac{25 \pm \sqrt{81}}{2} \in \{17, 8\}$. Two solutions:

$$z = 6 + 17i \text{ or } z = 6 + 8i.$$

A check: for $z = 6 + 8i$, $|z - 12| = |-6 + 8i| = 10$ and $|z - 8i| = |6| = 6$, ratio $5/3$ as required. □

What to take away. Every equation in $|z - a|$ is a circle or a line in the Argand plane. A system of two such equations is the intersection of two loci, solved by translating each to real algebra in x and y. The geometric picture (perpendicular bisector, Apollonian circle) is a useful cross-check, but the computation is ordinary real algebra.

Problem 10.2. Show that every complex number $a + bi$ with $a^2 + b^2 = 1$ and $b \neq 0$ can be represented as

$$a + bi \;=\; \frac{c + i}{c - i}$$

for some real number c.

Technique. Solve the equation for c in terms of a and b; verify the candidate satisfies the required constraint using $a^2 + b^2 = 1$.

Discussion. Rearrange $a + bi = \frac{c+i}{c-i}$ by multiplying through by $c - i$:

$$(a + bi)(c - i) \;=\; c + i.$$

Expand the left side: $ac + b - i(a - bc) = c + i$. Matching real and imaginary parts,

$$ac + b \;=\; c \qquad \text{and} \qquad bc - a \;=\; 1.$$

The first gives $c(1 - a) = b$, so $c = \frac{b}{1-a}$, provided $a \neq 1$ (which follows from $b \neq 0$ together with $a^2 + b^2 = 1$). The second gives $c = \frac{a+1}{b}$.

Consistency. The two expressions for c must agree. Cross-multiplying, $b \cdot b = (1 - a)(a + 1)$, that is,

$$b^2 \;=\; 1 - a^2,$$

which is exactly the hypothesis $a^2 + b^2 = 1$. So the two definitions of c coincide, and the real number

$$c \;=\; \frac{a + 1}{b}$$

furnishes the required representation. □

What to take away. Every point on the unit circle except $+1$ is the image of exactly one real parameter c under the Möbius transformation $c \mapsto \frac{c+i}{c-i}$: the point -1 is reached at $c = 0$, while $+1$ arises only in the limit $c \to \pm\infty$. (The problem sets aside both real-axis points ± 1 by asking for $b \neq 0$.) This is the reverse of the stereographic projection through i: real c is the point on the x-axis whose image under the transformation is $a + bi$. The identity $b^2 = 1 - a^2$ is not decorative; it is what makes two a-priori-different values of c agree.

De Moivre and the Chebyshev recurrence

Problem 10.3. Define a sequence of polynomials by $T_0(x) = 1$, $T_1(x) = x$, and

$$T_{n+1}(x) \;=\; 2x\,T_n(x) - T_{n-1}(x) \qquad (n \geq 1).$$

Prove that $\cos(n\theta) = T_n(\cos\theta)$ for every real θ and every $n \geq 0$.

Technique. Strong induction on n, using the product-to-sum identity $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta\cos(n\theta)$.

Discussion. Base cases: $\cos 0 = 1 = T_0(\cos\theta)$, and $\cos\theta = T_1(\cos\theta)$. Both hold.

For the step, let $n \geq 1$ and assume $\cos(k\theta) = T_k(\cos\theta)$ for all $k \leq n$. The product-to-sum identity gives

$$\cos((n+1)\theta) \;=\; 2\cos\theta\cos(n\theta) - \cos((n-1)\theta).$$

Substitute the inductive hypothesis on the right:

$$\cos((n+1)\theta) \;=\; 2\cos\theta \cdot T_n(\cos\theta) - T_{n-1}(\cos\theta),$$

which equals $T_{n+1}(\cos\theta)$ by the recurrence defining T_{n+1}. The step closes. □

What to take away. The polynomial T_n is the n-th Chebyshev polynomial of the first kind. The recurrence $T_{n+1} = 2xT_n - T_{n-1}$ is a polynomial identity, not a trigonometric one; but its evaluation at $x = \cos\theta$ reproduces the trigonometric recurrence. This is the standard bridge from trigonometry to polynomial identities: every expression $\cos(n\theta)$ is a fixed polynomial in $\cos\theta$, computable by the recurrence.

Problem 10.4. Suppose the complex number z satisfies $z + \frac{1}{z} = 2\cos\theta$ for some real θ. Find a closed-form expression for $z^n + \frac{1}{z^n}$ in terms of n and θ, for every positive integer n.

Technique. Identify z as one of $e^{\pm i\theta}$; apply De Moivre to compute $z^n + 1/z^n$.

Discussion. The hypothesis $z + 1/z = 2\cos\theta$ is a quadratic in z:

$$z^2 - 2\cos\theta \cdot z + 1 \;=\; 0.$$

By the quadratic formula, $z = \cos\theta \pm i\sqrt{1-\cos^2\theta} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$.

By De Moivre, $z^n = e^{\pm in\theta}$ and $1/z^n = e^{\mp in\theta}$. Therefore

$$z^n + \frac{1}{z^n} \;=\; e^{in\theta} + e^{-in\theta} \;=\; 2\cos(n\theta).$$

The result is independent of which square root was chosen for z. □

What to take away. The substitution $z = e^{i\theta}$ turns the algebraic identity $z^n + 1/z^n = 2\cos(n\theta)$ into a tautology. Conversely: every polynomial identity in $z + 1/z$ corresponds, under this substitution, to a trigonometric identity in $\cos\theta$. Problem 10.3 is the polynomial side of this correspondence; this problem is the exponential side.

Roots of unity

Problem 10.5. For every integer $n \geq 2$, prove that

$$\prod_{k=1}^{n-1} \sin\frac{k\pi}{n} \;=\; \frac{n}{2^{n-1}}.$$

Technique. Factor $z^n - 1$ using the n-th roots of unity; divide by $(z-1)$; evaluate at $z = 1$; take moduli.

Discussion. Let $\zeta = e^{2\pi i/n}$, a primitive n-th root of unity. The polynomial $z^n - 1$ factors over $\mathbb{C}$ as

$$z^n - 1 \;=\; \prod_{k=0}^{n-1} (z - \zeta^k).$$

Divide both sides by $z - 1$ (which equals the $k = 0$ factor):

$$\frac{z^n - 1}{z - 1} \;=\; \prod_{k=1}^{n-1} (z - \zeta^k).$$

The left-hand side is the polynomial $1 + z + z^2 + \cdots + z^{n-1}$, whose value at $z = 1$ is n. So

$$\prod_{k=1}^{n-1} (1 - \zeta^k) \;=\; n.$$

Take moduli and use $|1 - e^{i\phi}| = 2|\sin(\phi/2)|$:

$$\prod_{k=1}^{n-1} |1 - \zeta^k| \;=\; \prod_{k=1}^{n-1} 2 \sin \frac{k\pi}{n} \;=\; n,$$

since $\sin(k\pi/n) > 0$ for $k = 1, \ldots, n-1$. Dividing by the factor 2^{n-1},

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} \;=\; \frac{n}{2^{n-1}},$$

as claimed. □

What to take away. A purely trigonometric identity has a purely algebraic proof via roots of unity: write down the polynomial factorisation of $z^n - 1$, strip the trivial root, and take moduli at $z = 1$. The identity $|1 - e^{i\phi}| = 2|\sin(\phi/2)|$ is the bridge, converting complex magnitudes to real sines. The technique generalises: every product of sines with arguments at evenly spaced angles can be evaluated this way.

Problem 10.6. Find all complex roots of $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$, and factor this polynomial over $\mathbb{R}$.

Technique. Recognise the sum as a finite geometric series; identify the roots as the non-trivial 6th roots of unity; pair conjugates to obtain real quadratic factors.

Discussion. The sum telescopes under the identity

$$z^5 + z^4 + z^3 + z^2 + z + 1 \;=\; \frac{z^6 - 1}{z - 1},$$

valid for $z \neq 1$. The roots of the left-hand side are therefore the 6th roots of unity other than 1:

$$\zeta^k \;=\; e^{ik\pi/3}, \qquad k = 1, 2, 3, 4, 5.$$

Pair conjugate roots: ζ with $\zeta^5 = \bar{\zeta}$, ζ^2 with $\zeta^4 = \overline{\zeta^2}$, and $\zeta^3 = -1$ (its own conjugate). Each conjugate pair gives a real quadratic factor via

$$(z - \zeta^k)(z - \overline{\zeta^k}) \;=\; z^2 - 2\cos\tfrac{k\pi}{3} \cdot z + 1.$$

With $\cos(\pi/3) = 1/2$ and $\cos(2\pi/3) = -1/2$, the two quadratic factors are $z^2 - z + 1$ and $z^2 + z + 1$. The real root -1 contributes the linear factor $z + 1$. Combining,

$$z^5 + z^4 + z^3 + z^2 + z + 1 \;=\; (z + 1)(z^2 - z + 1)(z^2 + z + 1).$$

□

What to take away. The sum $1 + z + \cdots + z^{n-1}$ is $(z^n - 1)/(z - 1)$, a finite geometric series. Its roots are the non-trivial n-th roots of unity, distributed symmetrically around the unit circle. Pairing conjugate roots produces the real factorisation in one line; the real factorisation of the sum polynomial for $n = 6$ is $(z + 1)(z^2 - z + 1)(z^2 + z + 1)$, and an analogous factorisation holds for every n.

Applications to real polynomials

Problem 10.7. Factor the polynomial $x^4 + 1$ as a product of two real quadratic polynomials.

Technique. Find the four complex roots of $x^4 + 1 = 0$; pair conjugates; multiply out each pair.

Discussion. The roots satisfy $x^4 = -1 = e^{i\pi}$, hence $x = e^{i(\pi+2\pi k)/4}$ for $k = 0, 1, 2, 3$. Explicitly,

$$x \in \{e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}\}.$$

Conjugate pairs: $e^{i\pi/4}$ with $e^{i7\pi/4} = e^{-i\pi/4}$, and $e^{i3\pi/4}$ with $e^{i5\pi/4} = e^{-i3\pi/4}$.

The first pair gives the real quadratic factor

$$(x - e^{i\pi/4})(x - e^{-i\pi/4}) \;=\; x^2 - 2\cos\tfrac{\pi}{4}\cdot x + 1 \;=\; x^2 - \sqrt{2}\,x + 1,$$

and the second pair gives $x^2 + \sqrt{2}\,x + 1$ (since $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$). So

$$x^4 + 1 \;=\; (x^2 - \sqrt{2}\,x + 1)(x^2 + \sqrt{2}\,x + 1).$$

As a check, the product equals $(x^2+1)^2 - (\sqrt{2}\,x)^2 = x^4 + 2x^2 + 1 - 2x^2 = x^4 + 1$. □

What to take away. A polynomial that has no rational roots can still factor over $\mathbb{R}$ as a product of quadratics; the key is to pass through $\mathbb{C}$ to find the roots, then pair conjugates. Such a factor, coming from a non-real conjugate pair $p \pm qi$ with $q \neq 0$, is $x^2 - 2px + (p^2 + q^2)$, whose discriminant is $4p^2 - 4(p^2 + q^2) = -4q^2 < 0$. A negative discriminant is exactly the signature of non-real roots; a non-negative one would mean the factor split further over $\mathbb{R}$.

Problem 10.8. Let $P(x)$ be a polynomial with real coefficients. (a) Prove that if $\alpha = a + bi$ with $b \neq 0$ is a root of P, then $\bar{\alpha} = a - bi$ is also a root. (b) Deduce that every real polynomial of odd degree has at least one real root.

Technique. (a) Use that complex conjugation is a ring homomorphism: $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, and $\bar{r} = r$ for real r. (b) Count roots with multiplicity.

Discussion. (*a*) Write $P(x) = \sum_{k=0}^{n} c_k x^k$ with every $c_k \in \mathbb{R}$. Conjugation distributes over sums and products, so

$$\overline{P(\alpha)} \;=\; \sum_k \overline{c_k \alpha^k} \;=\; \sum_k \bar{c}_k \bar{\alpha}^k \;=\; \sum_k c_k \bar{\alpha}^k \;=\; P(\bar{\alpha}),$$

using $\bar{c}_k = c_k$ for real c_k in the third step. If $P(\alpha) = 0$, then $\overline{P(\alpha)} = 0$, so $P(\bar{\alpha}) = 0$, i.e. $\bar{\alpha}$ is a root.

(b) Let P have odd degree n. By the fundamental theorem of algebra, P has n roots in $\mathbb{C}$, counted with multiplicity. By part (a), the non-real roots pair into conjugate pairs; each pair contributes two to the total count. The total count of non-real roots is therefore even. The remaining roots are real. Since n is odd, the total count is odd, and the number of real roots is odd minus even, which is odd, and in particular at least one. □

What to take away. Complex conjugation is a ring homomorphism on $\mathbb{C}$, and real polynomials respect it. The two consequences in this problem, conjugate-root pairing and the existence of a real root in odd degree, are the standard machinery for real polynomial factorisation. The same pairing drove the factorisations in Problems 10.7 and 10.6.

CLASSICAL IDENTITIES

PROBLEM 10.9. For real α with $\alpha \neq 2k\pi$ for any integer k, and for every positive integer n, prove that

$$\sum_{k=0}^{n-1} \cos(k\alpha) \;=\; \frac{\sin(n\alpha/2)\cdot\cos((n-1)\alpha/2)}{\sin(\alpha/2)}.$$

Technique. Express $\cos(k\alpha)$ as the real part of $e^{ik\alpha}$; sum the resulting geometric series; take real parts.

Discussion. The finite geometric series

$$\sum_{k=0}^{n-1} e^{ik\alpha} \;=\; \frac{e^{in\alpha}-1}{e^{i\alpha}-1}$$

is valid because $e^{i\alpha} \neq 1$ (the hypothesis $\alpha \neq 2k\pi$). Factor both numerator and denominator by the half-angle exponentials:

$$e^{in\alpha}-1 \;=\; e^{in\alpha/2}(e^{in\alpha/2}-e^{-in\alpha/2}) \;=\; 2ie^{in\alpha/2}\sin(n\alpha/2),$$

and similarly $e^{i\alpha} - 1 = 2ie^{i\alpha/2}\sin(\alpha/2)$. Dividing,

$$\sum_{k=0}^{n-1} e^{ik\alpha} \;=\; e^{i(n-1)\alpha/2} \cdot \frac{\sin(n\alpha/2)}{\sin(\alpha/2)}.$$

Taking real parts of both sides gives the claimed identity. □

What to take away. Every finite trigonometric sum over evenly spaced angles is a finite geometric series in $e^{i\alpha}$ in disguise. The half-angle factoring is the mechanical step; taking real (or imaginary) parts delivers the closed form. The sister identity for $\sum \sin(k\alpha)$ follows by taking imaginary parts of the same calculation.

Problem 10.10. Let z_1, z_2 be complex numbers with non-zero imaginary parts. Prove that if $z_1 + z_2$ and $z_1 z_2$ are both real, then $z_2 = \bar{z}_1$.

Technique. Recognise z_1, z_2 as the roots of a real quadratic; apply the conjugate-pair theorem of Problem 10.8.

Discussion. Consider the quadratic polynomial

$$Q(z) \;=\; z^2 - (z_1 + z_2)z + z_1 z_2.$$

By construction its roots are z_1 and z_2. By hypothesis, both coefficients $-(z_1 + z_2)$ and $z_1 z_2$ are real, so Q has real coefficients.

By the conjugate-root pairing of Problem 10.8(a), since z_1 is a root of Q with non-zero imaginary part, $\bar{z}_1$ is also a root. A quadratic has at most two roots, so $\{z_1, z_2\} \subseteq \{z_1, \bar{z}_1\}$. Since $z_1 \neq \bar{z}_1$ (because z_1 has non-zero imaginary part), the only way is $z_2 = \bar{z}_1$. □

What to take away. The sum and product of two complex numbers are exactly the coefficients of the monic quadratic having them as roots. Realness of the sum and product therefore amounts to realness of the quadratic, and the conjugate-pair theorem does the rest. This is the quickest route from the hypothesis to the conclusion and a model for how Vieta's formulas and conjugate pairing collaborate.

Chapter 11

Sequences, series, and recurrences

Arithmetic and geometric progressions

Problem 11.1. The sequence $a_1, a_2, a_3, \ldots$ is an arithmetic progression. It is given that $a_m = n$ and $a_n = m$, where $m \neq n$ are positive integers. Find a_{m+n}.

Technique. Write the AP as $a_k = a_1 + (k-1)d$; the two given conditions determine a_1 and d; substitute to find the desired term.

Discussion. From $a_k = a_1 + (k-1)d$,

$$a_m = a_1 + (m-1)d = n, \qquad a_n = a_1 + (n-1)d = m.$$

Subtracting, $(m-n)d = n-m$, so $d = -1$ (using $m \neq n$). Substituting back into the first equation, $a_1 = n - (m-1)(-1) = n + m - 1$. Hence

$$a_{m+n} = a_1 + (m+n-1)d = (n+m-1) + (m+n-1)(-1) = 0.$$

□

What to take away. Two linear conditions on a_1 and d determine the entire progression. The "symmetric" hypothesis $a_m = n$, $a_n = m$ is designed so that a subtraction eliminates a_1 and exposes the common difference d as a signed ratio. The conclusion $a_{m+n} = 0$ is the signature of this kind of symmetry, and a good instinct for what such a question is asking.

Problem 11.2. Let $a_1, a_2, \dots, a_n$ be an arithmetic progression of positive reals with common difference $d \neq 0$. Prove that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

Technique. Partial-fraction telescoping: express $\frac{1}{a_k a_{k+1}}$ as a difference of reciprocals whose successive terms cancel.

Discussion. For consecutive AP terms a_k, a_{k+1} the difference is $a_{k+1} - a_k = d$. Therefore

$$\frac{1}{a_k} - \frac{1}{a_{k+1}} = \frac{a_{k+1} - a_k}{a_k a_{k+1}} = \frac{d}{a_k a_{k+1}},$$

which rearranges to

$$\frac{1}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{1}{a_k} - \frac{1}{a_{k+1}}\right).$$

Summing over $k = 1, 2, \dots, n-1$, the right-hand side telescopes:

$$\sum_{k=1}^{n-1} \frac{1}{a_k a_{k+1}} = \frac{1}{d}\left(\frac{1}{a_1} - \frac{1}{a_n}\right) = \frac{a_n - a_1}{d\, a_1 a_n}.$$

The numerator $a_n - a_1 = (n-1)d$, and the factors of d cancel, leaving $(n-1)/(a_1 a_n)$, as claimed. □

What to take away. Whenever a sum has the shape $\sum f(k) g(k+1)$ with f, g linear, the ratio can be split into a telescoping difference. Here the split $\frac{1}{a_k a_{k+1}} = \frac{1}{d}(\frac{1}{a_k} - \frac{1}{a_{k+1}})$ is the only trick; everything else is cancellation. The shape of the right-hand side, "count-over-endpoint-product," is the diagnostic to look for.

Telescoping

Problem 11.3. Let m be a fixed positive integer. Prove that for every positive integer n,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)\cdots(k+m)} = \frac{1}{m}\left(\frac{1}{m!} - \frac{n!}{(n+m)!}\right).$$

Technique. Express the summand as $(1/m)$ times the finite difference of a simpler rational function, then telescope.

Discussion. Define $f(k) = \frac{1}{k(k+1)\cdots(k+m-1)}$, a product of m consecutive factors starting at k. Then $f(k)$ and $f(k+1)$ share $m-1$ denominator factors; their difference, brought to the common denominator $k(k+1)\cdots(k+m)$, is

$$f(k) - f(k+1) = \frac{(k+m)-k}{k(k+1)\cdots(k+m)} = \frac{m}{k(k+1)\cdots(k+m)}.$$

Therefore

$$\frac{1}{k(k+1)\cdots(k+m)} = \frac{1}{m}[f(k) - f(k+1)],$$

and summing from $k=1$ to n,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} = \frac{1}{m}[f(1) - f(n+1)].$$

Now $f(1) = \frac{1}{1\cdot 2\cdots m} = \frac{1}{m!}$, and $f(n+1) = \frac{1}{(n+1)(n+2)\cdots(n+m)} = \frac{n!}{(n+m)!}$. Substituting yields the claimed identity. □

What to take away. The base-case $m = 1$ is the familiar identity $\sum \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$. What is not immediately obvious is that the same telescoping works in a one-parameter family: shifting the length of the product by one extends the identity to every m. The trick $f(k) - f(k+1) = m/[k(k+1)\cdots(k+m)]$ is really a discrete derivative, and the telescoping is discrete integration.

Problem 11.4. Prove that for every positive integer n,

$$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + n\cdot n! = (n+1)! - 1.$$

Technique. Rewrite $k\cdot k!$ as a finite difference of factorials; the sum then telescopes.

Discussion. The identity $k = (k+1) - 1$ gives

$$k\cdot k! = (k+1)\cdot k! - k! = (k+1)! - k!.$$

Summing from $k=1$ to n, the right-hand side telescopes:

$$\sum_{k=1}^{n} k\cdot k! = \sum_{k=1}^{n}\big[(k+1)! - k!\big] = (n+1)! - 1!,$$

which is $(n+1)! - 1$. ☐

What to take away. The conversion $k \cdot k! = (k+1)! - k!$ is the discrete analogue of the integration-by-parts identity for ke^k. Every sum whose summand involves a polynomial times a factorial collapses the same way: isolate the factorial growth, wedge the polynomial factor between consecutive factorials, telescope. The closed form $(n+1)! - 1$ is worth committing to memory; it is the finite-factorial cousin of the geometric series.

LINEAR RECURRENCES

PROBLEM 11.5. The Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and the recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Derive the closed form

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \qquad \varphi = \frac{1+\sqrt{5}}{2},\ \psi = \frac{1-\sqrt{5}}{2}.$$

Technique. Look for solutions of the form $F_n = r^n$; the recurrence forces r to satisfy a quadratic; the two roots φ, ψ give two independent solutions whose linear combination matches the initial conditions.

Discussion. Suppose $F_n = r^n$ is a solution of $F_{n+1} = F_n + F_{n-1}$. Substituting and dividing by r^{n-1},

$$r^2 = r + 1, \qquad \text{that is,} \qquad r^2 - r - 1 = 0.$$

This is the *characteristic equation* of the recurrence. Its roots are $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Both φ^n and ψ^n solve the recurrence, and because the recurrence is linear, every linear combination $A\varphi^n + B\psi^n$ is also a solution.

Fit the initial conditions. For $n = 0$, $A + B = 0$, so $B = -A$. For $n = 1$, $A\varphi - A\psi = 1$, i.e. $A(\varphi - \psi) = 1$. Since $\varphi - \psi = \sqrt{5}$, we get $A = 1/\sqrt{5}$. Therefore

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

A uniqueness note: a two-term linear recurrence with given F_0 and F_1 has a unique solution, so any formula agreeing at $n = 0, 1$ and satisfying the recurrence must be correct for every n. □

What to take away. The leap from recurrence to closed form goes through the characteristic equation. Solutions of the recurrence are a two-dimensional vector space over $\mathbb{R}$; the two roots of the quadratic produce a basis; the initial conditions pick out coordinates. The appearance of $\sqrt{5}$ in a formula for integers F_n is striking and points to a deeper fact: ψ^n is small (since $|\psi| < 1$), so F_n is the nearest integer to $\varphi^n/\sqrt{5}$.

Problem 11.6. The sequence $a_0, a_1, a_2, \ldots$ is defined by $a_0 = 1$, $a_1 = 4$, and the recurrence $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2$. Find a closed form for a_n.

Technique. Solve the characteristic equation; recognise the double root; adjoin a second independent solution of the form nr^n; match initial conditions.

Discussion. The characteristic equation is $r^2 - 4r + 4 = 0$, that is, $(r-2)^2 = 0$, with the single repeated root $r = 2$. This supplies one family of solutions $a_n = r^n = 2^n$, but a two-term recurrence has a two-dimensional solution space; one family is not enough.

Try $a_n = n \cdot 2^n$ as a second candidate. Substitute into the recurrence:

$$4a_{n-1} - 4a_{n-2} = 4(n-1)2^{n-1} - 4(n-2)2^{n-2}.$$

Factor out 2^{n-2}:

$$4a_{n-1} - 4a_{n-2} = 2^{n-2}[8(n-1) - 4(n-2)] = 2^{n-2} \cdot 4n = n \cdot 2^n.$$

The last expression equals the proposed a_n, so $a_n = n \cdot 2^n$ is indeed a solution. We now have two independent solutions 2^n and $n \cdot 2^n$.

The general solution is $a_n = (A + Bn) \cdot 2^n$. Fit $a_0 = 1$: $A = 1$. Fit $a_1 = 4$: $(1 + B) \cdot 2 = 4$, so $B = 1$. Therefore

$$a_n = (n+1) \cdot 2^n.$$

□

What to take away. When the characteristic polynomial has a repeated root r, the two candidate solutions r^n and r^n coincide and span only a one-dimensional space. The second independent solution is $n \cdot r^n$. This is the exact discrete analogue of the fact that e^{rx} and $x\mathrm{e}^{rx}$ solve a second-order linear differential equation with a double characteristic root r. The pattern generalises: a root of multiplicity k contributes k independent solutions $r^n, n \cdot r^n, \dots, n^{k-1} \cdot r^n$.

ARITHMETICO-GEOMETRIC SUMS AND PROGRESSIONS IN DISGUISE

PROBLEM 11.7. For $r \neq 1$ and every positive integer n, find a closed form for

$$S_n \;=\; \sum_{k=1}^{n} k\, r^k \;=\; r + 2r^2 + 3r^3 + \cdots + nr^n.$$

Technique. Multiply the sum by r and subtract; the term-by-term cancellation converts the weights k into constant 1s, reducing the problem to a finite geometric series.

Discussion. Write

$$r\, S_n \;=\; r^2 + 2r^3 + 3r^4 + \cdots + nr^{n+1}.$$

Subtract this from S_n:

$$S_n - rS_n \;=\; r + (2-1)r^2 + (3-2)r^3 + \cdots + (n-(n-1))r^n - nr^{n+1}.$$

The weighted differences all collapse to 1, leaving

$$(1-r)S_n \;=\; r + r^2 + r^3 + \cdots + r^n - nr^{n+1}.$$

The remaining geometric sum is $r(1-r^n)/(1-r)$, so

$$(1-r)S_n \;=\; \frac{r(1-r^n)}{1-r} - nr^{n+1}.$$

Dividing by $(1-r)$,

$$S_n \;=\; \frac{r - (n+1)r^{n+1} + nr^{n+2}}{(1-r)^2}.$$

The numerator follows by placing the two terms on the right-hand side over the common denominator $(1-r)$ and expanding. □

What to take away. The "multiply by the ratio and subtract" move is the universal tool for arithmetico-geometric sums (AP times GP). It reduces the weighted series to a plain geometric series, which has its own closed form. The same trick evaluates $\sum k^2 r^k$ (apply the move twice) and, more generally, $\sum p(k) r^k$ for any polynomial p.

Problem 11.8. Let $a_1, a_2, a_3, \ldots$ be an arithmetic progression, and for each positive integer n let $S_n = a_1 + a_2 + \cdots + a_n$ be its n-th partial sum. Prove that

$$S_{3n} \;=\; 3(S_{2n} - S_n).$$

Technique. Observe that the three block sums S_n, $S_{2n} - S_n$, $S_{3n} - S_{2n}$ are themselves an arithmetic progression.

Discussion. Let the AP have common difference d. The three block sums are

$$B_1 = \sum_{k=1}^{n} a_k, \qquad B_2 = \sum_{k=n+1}^{2n} a_k, \qquad B_3 = \sum_{k=2n+1}^{3n} a_k.$$

Each block contains n consecutive AP terms. Term by term, every entry of B_2 exceeds the corresponding entry of B_1 by nd: $a_{n+k} = a_k + nd$ for $k = 1, \ldots, n$. Summing over k,

$$B_2 - B_1 \;=\; \sum_{k=1}^{n} nd \;=\; n^2 d.$$

The same argument, applied to the pair (B_2, B_3), gives $B_3 - B_2 = n^2 d$. So the three quantities B_1, B_2, B_3 form an arithmetic progression with common difference $n^2 d$.

For any three-term AP B_1, B_2, B_3, the middle term is the average of the outer two: $B_1 + B_3 = 2B_2$, that is, $B_3 = 2B_2 - B_1$. Translate back using $B_1 = S_n$, $B_2 = S_{2n} - S_n$, $B_3 = S_{3n} - S_{2n}$:

$$S_{3n} - S_{2n} \;=\; 2(S_{2n} - S_n) - S_n,$$

and rearranging gives $S_{3n} \;=\; 3S_{2n} - 3S_n \;=\; 3(S_{2n} - S_n)$, as claimed. □

What to take away. The proof never computes S_n in closed form; it exploits only the structural fact that each block of n consecutive AP terms beats the previous block by a fixed shift. The same idea shows that if every block of length n has an equal number of terms, the block sums form an AP of common difference n^2d. A surprising amount of AP lore reduces to recognising these block-level regularities.

CLASSICAL IDENTITIES FOR GEOMETRIC PROGRESSIONS

PROBLEM 11.9. Let $b_1, b_2, \dots, b_n$ be a geometric progression of positive reals with common ratio $q \neq 1$. Let

$$P = b_1 b_2 \cdots b_n, \qquad S = \sum_{k=1}^{n} b_k, \qquad T = \sum_{k=1}^{n} \frac{1}{b_k}$$

denote respectively the product of the terms, the sum of the terms, and the sum of their reciprocals. Prove that $P^2 = (S/T)^n$, equivalently $P = (S/T)^{n/2}$.

Technique. Write $b_k = b_1 q^{k-1}$ and compute P, S, and T in closed form; then compare P^2 with $(S/T)^n$ directly.

Discussion. With $b_k = b_1 q^{k-1}$, the product is

$$P \;=\; b_1^n \cdot q^{0+1+\cdots+(n-1)} \;=\; b_1^n \, q^{n(n-1)/2},$$

so $P^2 = b_1^{2n} \, q^{n(n-1)}$.

The sum S is the standard geometric-series formula $S = b_1(q^n - 1)/(q-1)$. The reciprocals form another GP, with first term $1/b_1$ and common ratio $1/q$, so

$$T \;=\; \frac{1}{b_1} \cdot \frac{(1/q)^n - 1}{(1/q) - 1}.$$

Multiplying numerator and denominator of the fraction by q^n to clear reciprocals,

$$T \;=\; \frac{1}{b_1} \cdot \frac{1 - q^n}{q^{n-1}(1-q)} \;=\; \frac{1}{b_1 \, q^{n-1}} \cdot \frac{q^n - 1}{q - 1}.$$

Therefore

$$\frac{S}{T} = \frac{b_1(q^n-1)/(q-1)}{(q^n-1)/[b_1q^{n-1}(q-1)]} = b_1^2q^{n-1}.$$

Raising to the n-th power,

$$(S/T)^n = b_1^{2n}q^{n(n-1)} = P^2,$$

as required. □

What to take away. The three quantities P, S, T attached to a GP are not independent: any two of them determine the third, and the clean relation $P^2 = (S/T)^n$ is the witness. The mechanism is that the reciprocal GP has the same length and common-ratio-up-to-inversion as the original; hence S/T collapses to $b_1^2q^{n-1}$, which is the geometric-mean square of the first and last terms. The identity generalises: for any symmetric function that depends only on multiplicative structure, the same dual relationship to the reciprocal sequence holds.

Problem 11.10. Let $a_1 < a_2 < a_3 < \cdots$ be an arithmetic progression and $b_1 < b_2 < b_3 < \cdots$ a geometric progression of positive reals. Suppose the first two terms agree: $a_1 = b_1 > 0$ and $a_2 = b_2$. Prove that for every integer $k \geq 3$,

$$b_k > a_k.$$

Technique. Express a_k and b_k in terms of the shared ratio q; reduce the inequality to Bernoulli's inequality $(1+x)^n > 1 + nx$ for $x > 0$ and $n \geq 2$; prove Bernoulli from the binomial expansion.

Discussion. Let $b_1 = a_1 = a > 0$, and let q be the common ratio of the GP, so $b_2 = aq$. The matching condition $a_2 = b_2$ forces the AP's common difference to be $d = b_2 - b_1 = aq - a = a(q-1)$. Because the GP is strictly increasing and $a > 0$, we have $q > 1$, hence $d > 0$ as well.

For $k \geq 3$, the k-th terms are

$$a_k = a+(k-1)d = a+(k-1)a(q-1) = a[1+(k-1)(q-1)],$$

$$b_k = aq^{k-1}.$$

Dividing by $a > 0$, the inequality $b_k > a_k$ reduces to $q^{k-1} > 1 + (k-1)(q-1)$. Set $x = q - 1 > 0$ and $n = k - 1 \geq 2$; the inequality becomes

$$(1+x)^n \; > \; 1 + nx.$$

This is Bernoulli's inequality for integer $n \geq 2$ and positive x.

Bernoulli's inequality, proved inline. The binomial theorem gives, for integer $n \geq 0$,

$$(1+x)^n \; = \; \sum_{j=0}^{n} \binom{n}{j} x^j \; = \; 1 + nx + \sum_{j=2}^{n} \binom{n}{j} x^j.$$

For $n \geq 2$ and $x > 0$, the tail sum is strictly positive, because $\binom{n}{2}x^2 > 0$ and every subsequent term is non-negative. Therefore $(1+x)^n > 1 + nx$, which completes the proof. $\square$

What to take away. Given the same first two terms, a GP with ratio $q > 1$ pulls ahead of an AP with matching difference almost immediately. The reason is exponential-beats-linear, but made finite and elementary by Bernoulli. The binomial-expansion proof of Bernoulli used here is the cleanest of the several available (induction on n, convexity, derivative arguments); it is worth reaching for whenever $(1+x)^n$ appears in an inequality.

Chapter 12

Counting and combinatorial arguments

Arrangements with constraints

Problem 12.1. How many subsets of $\{1, 2, \ldots, n\}$ contain no two consecutive integers?

Technique. Recursion on n: case-split on whether the largest element n belongs to the subset.

Discussion. Let a_n denote the count. Base values: $a_0 = 1$ (only the empty subset of $\emptyset$), and $a_1 = 2$ (the subsets $\emptyset$ and $\{1\}$).

For $n \geq 2$, partition the valid subsets of $\{1, \ldots, n\}$ by whether n is included.

- If $n \notin S$, then S is a valid subset of $\{1, \ldots, n-1\}$. Such subsets number a_{n-1}.
- If $n \in S$, then $n - 1 \notin S$ by the no-consecutive rule, so $S \setminus \{n\}$ is a valid subset of $\{1, \ldots, n-2\}$. Such subsets number a_{n-2}.

Adding, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

Define the Fibonacci sequence by $F_1 = F_2 = 1$ and $F_{m+1} = F_m + F_{m-1}$ for $m \geq 2$. The identities $a_0 = 1 = F_2$ and $a_1 = 2 = F_3$ match the base cases, and the recurrence is the same. By induction on n,

$$a_n = F_{n+2}.$$

□

What to take away. The technique is pure case analysis on the tail element, and the Fibonacci recurrence is not assumed but discovered: the constraint "no two consecutive" is what produces the Fibonacci structure. The same decomposition solves a family of "forbid adjacent" problems (tilings of a $1 \times n$ strip with monominoes and dominoes is the same count, by a one-line bijection).

Problem 12.2. Five men and five women are to be seated around a round table, with no two men sitting next to each other. How many distinct seatings are there, if seatings that differ only by rotation are considered the same?

Technique. Seat the women first in a circle (fixing rotational symmetry), then place the men one per gap between adjacent women.

Discussion. Treat the five women as the primary circular arrangement. Fixing one woman's seat to remove rotational symmetry, the remaining four women can be arranged in the other four women-seats in 4! ways.

Once the women are seated, there are exactly five gaps between adjacent women around the circle. The no-two-men-adjacent condition forces each gap to receive at most one man, and since there are five men and five gaps, each gap receives exactly one. The men can be assigned to the five gaps in 5! ways.

Total seatings: $4! \cdot 5! = 24 \cdot 120 = 2880$. □

What to take away. Two techniques combine in this problem. The first is *circle-fixing*: a circular arrangement of m objects with rotational symmetry has $(m-1)!$ distinct orderings, because any one object can be nailed to a reference seat. The second is the *gap method*: a forbid-adjacency condition on a second group is resolved by placing them into the gaps between the first group. Circle-fix first, then gap-fill.

SELECTIONS AND COMPOSITIONS

PROBLEM 12.3. For positive integers n and k, find the number of solutions in non-negative integers to the equation

$$x_1 + x_2 + \cdots + x_k \;=\; n.$$

Technique. Encode each solution as a sequence of n stars and $k-1$ bars; count the encodings.

Discussion. Represent a solution $(x_1, \ldots, x_k)$ by writing x_1 stars, then a bar, then x_2 stars, then a bar, and so on, ending with x_k stars. The encoding is a string of n stars and $k-1$ bars, total length $n+k-1$.

Conversely, any such string decodes to a unique solution: the i-th group of stars (between the $(i-1)$-th and i-th bar, where the 0-th and k-th bars denote the ends of the string) gives x_i. The encoding is therefore a bijection.

The number of strings of length $n+k-1$ with exactly $k-1$ bars is the number of ways to choose the positions of the bars:

$$\binom{n+k-1}{k-1}.$$

□

What to take away. Stars-and-bars converts a counting problem on solutions into a counting problem on binary strings. The bijection is the insight; everything else is a straight binomial coefficient. The same encoding counts multisets: the number of multisets of size n drawn from a k-element alphabet is the same $\binom{n+k-1}{k-1}$, for identical reasons.

PROBLEM 12.4. Let $n \geq 1$ and $1 \leq k \leq n-1$. Prove Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by a combinatorial argument, without invoking the factorial formula for binomial coefficients.

Technique. Partition the k-subsets of an n-set by the inclusion or exclusion of a chosen element.

Discussion. Let $[n] = \{1, 2, \ldots, n\}$. The left-hand side $\binom{n}{k}$ counts the k-element subsets of $[n]$. Partition these subsets by whether or not they contain the element n.

- Subsets containing n: removing n leaves a $(k-1)$-subset of $[n-1]$, and this removal is a bijection between the two sets. Count: $\binom{n-1}{k-1}$.
- Subsets not containing n: these are exactly the k-subsets of $[n-1]$. Count: $\binom{n-1}{k}$.

The two cases are mutually exclusive and together exhaust all k-subsets of $[n]$, so

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \qquad \square$$

What to take away. A binomial identity is always worth proving twice: algebraically via the factorial formula, and combinatorially via a bijection or a partition. The combinatorial proof here shows that Pascal's rule is a statement about subsets, not about algebraic manipulation. This is the model for Vieta, Vandermonde, and the rest of the classical binomial identities; in each case, the left-hand side and the right-hand side count the same finite set in two different ways.

DOUBLE COUNTING

PROBLEM 12.5. Let m, n, r be non-negative integers with $r \leq m + n$. Prove Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k},$$

by counting a single finite set in two ways.

Technique. Split the universe into two groups of sizes m and n, and count r-subsets by how many elements come from each group.

Discussion. Consider a set with $m + n$ elements, partitioned into a "left" group L of size m and a "right" group R of size n. The left-hand side counts the r-subsets of $L \cup R$ directly.

For the right-hand side, partition the r-subsets by the number k of elements drawn from L. A subset with exactly k elements from L and $r - k$ from R is specified by choosing the k left-elements ($\binom{m}{k}$ ways) and the $r - k$ right-elements ($\binom{n}{r-k}$ ways) independently, giving $\binom{m}{k}\binom{n}{r-k}$ subsets. Summing over all k from 0 to r (with $\binom{m}{k} = 0$ for $k > m$ and $\binom{n}{r-k} = 0$ for $r - k > n$, so only valid k contribute),

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}. \qquad \square$$

What to take away. Vandermonde's identity is the prototype of a *convolution* identity: the left-hand side is a count on the whole universe, and the right-hand side is a sum of counts on compatible splittings. The algebraic content is that multiplying generating functions $(1 + x)^m$ and $(1 + x)^n$ gives $(1 + x)^{m+n}$, and Vandermonde is the coefficient of x^r on both sides. In every convolution identity, the "whole" count equals the sum over splittings of the "piece" counts.

Problem 12.6. For integers $1 \le k \le n$, prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1},$$

by double-counting a natural combinatorial object.

Technique. Count pairs (committee, chair) in two orders: committee first, or chair first.

Discussion. A *committee-with-chair* on n people is a pair (C, c) where C is a k-element subset of $\{1, \dots, n\}$ and c is a distinguished element of C (the chair). Count such pairs two ways.

First counting: committee first. Pick the committee C ($\binom{n}{k}$ choices), then pick the chair c from within C (k choices). Total: $k\binom{n}{k}$.

Second counting: chair first. Pick the chair c from among all n people (n choices), then pick the other $k-1$ committee members from the remaining $n-1$ people ($\binom{n-1}{k-1}$ choices). Total: $n\binom{n-1}{k-1}$.

The two counts agree, so $k\binom{n}{k} = n\binom{n-1}{k-1}$. □

What to take away. Double counting is often the shortest proof of an identity involving binomial coefficients and a multiplicative factor. The move: find a combinatorial object whose count can be expressed two ways, each matching one side of the identity. Here, the factor k on the left and the factor n on the right point to "choose one distinguished element" applied to two different sets (the committee, or the whole universe), and the object (committee-with-chair) is the bridge.

INCLUSION-EXCLUSION

PROBLEM 12.7. A *derangement* of $\{1, 2, \ldots, n\}$ is a permutation π such that $\pi(i) \neq i$ for every i. Let D_n denote the number of derangements. Prove that

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Technique. Inclusion-exclusion over the events "position i is fixed".

Discussion. Let S denote the set of all permutations of $\{1, \ldots, n\}$ (size $n!$), and for each i let $A_i \subseteq S$ be the set of permutations with $\pi(i) = i$ (position i is a fixed point). A derangement is a permutation in none of the A_i, so

$$D_n = |S| - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

By the inclusion-exclusion principle,

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

The intersection $A_{i_1} \cap \cdots \cap A_{i_k}$ consists of permutations that fix the k specified positions; the remaining $n-k$ positions can be permuted arbitrarily, giving $(n-k)!$ such permutations. The number of k-subsets is $\binom{n}{k}$, so each k-sum evaluates to $\binom{n}{k}(n-k)!$.

Substituting and using $\binom{n}{k}(n-k)! = n!/k!$,

$$D_n \;=\; n! - \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!} \;=\; \sum_{k=0}^{n} \frac{(-1)^k n!}{k!} \;=\; n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

This is the claimed formula. □

What to take away. Inclusion-exclusion handles "none of the forbidden events occur" by signed-summing over which subsets of forbidden events *do* occur. The derangement count illustrates the method at its cleanest, because the intersection sizes depend only on the number of indices, not on which indices. As $n \to \infty$, $D_n/n!$ converges to $1/\mathrm{e}$, the first appearance of the exponential constant in combinatorics.

Problem 12.8. For a positive integer n with prime factorisation $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ (distinct primes p_i), let $\varphi(n)$ denote the number of integers in $\{1, 2, \ldots, n\}$ that are coprime to n. Prove Euler's formula

$$\varphi(n) \;=\; n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

Technique. Inclusion-exclusion over the events "divisible by p_i".

Discussion. Let $U = \{1, 2, \ldots, n\}$ and for each prime p_i define $B_i = \{k \in U : p_i \mid k\}$. An integer in U is coprime to n iff it is not divisible by any p_i, so

$$\varphi(n) \;=\; |U| - |B_1 \cup B_2 \cup \cdots \cup B_r|.$$

Intersection sizes: for any subset $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, r\}$, the set $B_{i_1} \cap \cdots \cap B_{i_s}$ is the set of multiples of $p_{i_1} p_{i_2} \cdots p_{i_s}$ (since the primes are distinct, the least common multiple equals the product). The number of multiples of a positive integer d in $\{1, \ldots, n\}$ is $\lfloor n/d \rfloor$, and when $d \mid n$ this simplifies to n/d. Here every $d = p_{i_1} \cdots p_{i_s}$ divides n, so

$$|B_{i_1} \cap \cdots \cap B_{i_s}| \;=\; \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_s}}.$$

Apply inclusion-exclusion:

$$\varphi(n) \;=\; n - \sum_i \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \cdots + (-1)^r \frac{n}{p_1 p_2 \cdots p_r}.$$

The right-hand side factors as

$$\varphi(n) \;=\; n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right),$$

because expanding the product reproduces the signed sum term by term. □

What to take away. Euler's totient formula is an inclusion-exclusion identity dressed up as a multiplicative one. The product form $\prod(1 - 1/p_i)$ reveals that φ is *multiplicative*: $\varphi(mn) = \varphi(m)\varphi(n)$ whenever $\gcd(m, n) = 1$. That multiplicativity is the engine behind most applications of φ in number theory, including Fermat-Euler's theorem $a^{\varphi(n)} \equiv 1 \pmod{n}$ for $\gcd(a, n) = 1$.

Pigeonhole and reflection

Problem 12.9. Prove that among any six people, there are either three who mutually know one another or three who are mutually strangers.

Technique. Colour the complete graph K_6 in two colours (red for "know", blue for "stranger"); apply the pigeonhole principle to the five edges emanating from one vertex; finish by case analysis on the monochromatic triangle.

Discussion. Model the six people as vertices of the complete graph K_6, and colour each edge red (if the two endpoints know each other) or blue (if they are strangers). The claim is that K_6 under any two-colouring contains a monochromatic triangle.

Fix a vertex v. It has five edges to the other vertices. By the pigeonhole principle, at least three of these five are the same colour, say red (the blue case is symmetric). Let a, b, c denote three vertices joined to v by red edges.

Now consider the three edges of the triangle $\{a, b, c\}$.

- If any one of them (say ab) is red, then the triangle vab has three red edges, a red monochromatic triangle.
- Otherwise, all three edges ab, bc, ca are blue, and the triangle abc is a blue monochromatic triangle.

In either case, K_6 contains a monochromatic triangle. □

What to take away. The number six is not arbitrary: the construction fails at five (there is a two-colouring of K_5 with no monochromatic triangle, namely the red 5-cycle plus blue diagonals). This problem is the base case of Ramsey's theorem, the statement that for every pair (r, s) there is a threshold $R(r, s)$ beyond which every red-blue colouring of K_n contains a red K_r or a blue K_s. The proof above shows $R(3, 3) \leq 6$; the cycle construction shows $R(3, 3) \geq 6$, so $R(3, 3) = 6$ exactly.

Problem 12.10. In an election, candidate A receives p votes and candidate B receives q votes, with $p > q > 0$. The votes are counted one at a time in some order. Prove that the number of counting orders in which candidate A is strictly ahead of B at every stage of the count is

$$\frac{p-q}{p+q}\binom{p+q}{p}.$$

Technique. Represent a counting order as a ± 1 lattice path; use the reflection principle to biject between "bad" paths starting with $+1$ and all paths starting with -1.

Discussion. Represent each counting order as a sequence of ± 1 steps: $+1$ for an A-vote, -1 for a B-vote. After t votes have been counted, the partial sum S_t equals (votes for A) minus (votes for B), so A is strictly ahead iff $S_t \geq 1$ for every $t \geq 1$. Let G denote the number of *good* orders, those for which $S_t \geq 1$ at every $t \geq 1$.

The total number of counting orders is $T = \binom{p+q}{p}$, obtained by choosing which p of the $p+q$ positions hold A-votes. A counting order is *bad* if $S_t = 0$ or $S_t < 0$ at some $t \geq 1$. Since $S_0 = 0$ and $S_1 \in \{1, -1\}$, "bad" splits into two sub-cases:

(i) orders starting with -1 (first vote is for B);

(ii) orders starting with $+1$ that later touch 0 (i.e., where $S_t = 0$ for some $t \geq 2$).

Reflection bijection. Consider an order of type (ii). Let τ be the smallest $t \geq 2$ with $S_\tau = 0$. Reflect the first τ steps: flip each ± 1 in positions $1, \ldots, \tau$ to its opposite. The new first step becomes -1, and the cumulative sum at step τ becomes $-0 = 0$ again; after step τ the subsequent steps are unchanged. The result is a valid counting order starting with -1, i.e., of type (i).

The map is invertible. Given any order of type (i), consider the first $t \geq 1$ with $S_t = 0$; such t exists because $S_{p+q} = p - q > 0$ and $S_1 = -1 < 0$, so by intermediate integer values the sum must first reach zero. Reflect the first t steps: the new order starts with $+1$ and touches 0 at step t, so it is of type (ii). This is the inverse of the previous map.

Therefore (type ii orders) and (type i orders) are in bijection, and have the same count. The number of orders starting with -1 (i.e., first vote for B) is the number of ways to arrange the remaining p A-votes and $q-1$ B-votes in $p+q-1$ positions:

$$|\text{type (i)}| = \binom{p+q-1}{p}.$$

Hence the total number of bad orders is

$$|\text{bad}| = |\text{type (i)}| + |\text{type (ii)}| = 2\binom{p+q-1}{p},$$

and

$$G = T - 2\binom{p+q-1}{p} = \binom{p+q}{p} - 2\binom{p+q-1}{p}.$$

Finally, using $\binom{p+q-1}{p} = \frac{q}{p+q}\binom{p+q}{p}$ (from the factorial forms, or from the committee-chairman identity applied with $n =$

$p + q, k = p$),

$$G = \binom{p+q}{p}\left(1 - \frac{2q}{p+q}\right) = \binom{p+q}{p} \cdot \frac{p-q}{p+q}. \qquad \square$$

What to take away. The reflection principle is the central bijection of combinatorial path-counting. Whenever a question asks for paths that stay above (or below) a line, the answer is usually “total paths minus (bijectively mapped) crossing paths.” The ballot problem is the original application, but the same reflection reappears in the derivation of the Catalan numbers (paths from origin to (n, n) that do not cross the diagonal) and in a hundred related questions. The trick is always to reflect the initial segment up to the first forbidden contact.

CHAPTER 13

The binomial theorem

THE THEOREM AND FIRST SUBSTITUTIONS

PROBLEM 13.1. Let n be a non-negative integer. Prove that

$$(1+x)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k,$$

where $\binom{n}{k}$ denotes the number of k-element subsets of an n-element set.

Technique. Expand the product $(1+x)(1+x)\cdots(1+x)$ by choosing, from each factor, either the 1 or the x; count the number of ways each monomial arises.

Discussion. Write the n-fold product as

$$(1+x)^n \;=\; (1+x)(1+x)\cdots(1+x).$$

When the product is expanded term by term, each term is the result of choosing, from each of the n factors, either the constant 1 or the variable x. The product of these choices contributes x^k exactly when x is selected from k of the n factors and 1 from the other $n-k$.

The number of ways to select which k factors contribute x is, by definition, $\binom{n}{k}$. Summing over all possible k from 0 to n,

$$(1+x)^n \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k. \qquad \square$$

What to take away. The binomial theorem is not an opaque algebraic identity; it is a counting statement about subsets, converted into a polynomial identity. Every subsequent result in this chapter, however elaborate, is obtained by substituting a specific x, applying a differential or integral operator, or multiplying two copies of the theorem together. The two simplest substitutions are immediate: setting $x = 1$ gives $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (total number of subsets of an n-set), and setting $x = -1$ gives, for $n \geq 1$, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$, i.e. the number of even-size subsets equals the number of odd-size subsets. Always try $x = \pm 1$ first when the theorem appears in a problem; the resulting identities are almost always useful.

Problem 13.2. For every positive integer n, prove that

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

Technique. Recognise the sum as the coefficient of x^n in a product of two binomial expansions, one differentiated and one plain; read off the closed form.

Discussion. Starting from $(1+x)^n = \sum_k \binom{n}{k}x^k$ and differentiating,

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}.$$

Multiply both sides by x:

$$nx(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k}.$$

Multiply this by the plain binomial expansion $(1+x)^n = \sum_j \binom{n}{j}x^j$:

$$nx(1+x)^{2n-1} = \left(\sum_k k\binom{n}{k}x^k\right)\left(\sum_j \binom{n}{j}x^j\right).$$

Compare the coefficient of x^n on both sides. On the left,

$$[x^n]\, nx(1+x)^{2n-1} = n\cdot[x^{n-1}](1+x)^{2n-1} = n\binom{2n-1}{n-1}.$$

On the right, the coefficient of x^n is

$$\sum_{k+j=n} k\binom{n}{k}\binom{n}{j} = \sum_{k=0}^{n} k\binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} k\binom{n}{k}^2,$$

using the symmetry $\binom{n}{n-k} = \binom{n}{k}$ in the last step. Equating,

$$\sum_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}. \qquad \square$$

What to take away. Two binomial expansions multiplied together produce convolution identities: the coefficient of x^r in $f(x)g(x)$ is $\sum_k [x^k]f \cdot [x^{r-k}]g$. By including a polynomial weight on one factor (here, the weight k, inserted by differentiating and multiplying by x), the family of identities extends arbitrarily far. Every identity of the convolutional form $\sum_k w(k)\binom{m}{k}\binom{n}{k}$ is a coefficient-comparison statement waiting to be read off a product of two binomial expansions.

Derivative and integral tricks

Problem 13.3. Find a closed form for

$$S_n = \sum_{k=0}^{n} k\binom{n}{k}.$$

Technique. Differentiate both sides of $(1+x)^n = \sum_k \binom{n}{k}x^k$ with respect to x, then evaluate at $x = 1$.

Discussion. Differentiate term by term:

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}x^{k-1}.$$

Evaluate at $x = 1$:

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k},$$

so

$$S_n = n\,2^{n-1}. \qquad \square$$

What to take away. Multiplying a binomial coefficient by k corresponds to differentiating the generating polynomial $(1+x)^n$ once. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ twice gives $\sum k(k-1)\binom{n}{k}x^{k-2} = n(n-1)(1+x)^{n-2}$, and evaluating at $x = 1$ yields $\sum k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$, from which $\sum k^2\binom{n}{k}$ follows by subtraction. Every polynomial weight on the binomial coefficients reduces to a finite combination of such derivative evaluations.

Problem 13.4. Find a closed form for

$$T_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

Technique. Integrate both sides of $(1+x)^n = \sum_k \binom{n}{k}x^k$ over $[0,1]$.

Discussion. On the left,

$$\int_0^1 (1+x)^n \,\mathrm{d}x = \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 = \frac{2^{n+1}-1}{n+1}.$$

On the right, integrating term by term,

$$\int_0^1 \sum_{k=0}^{n} \binom{n}{k}x^k \,\mathrm{d}x = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{k+1}.$$

Equating,

$$T_n = \frac{2^{n+1}-1}{n+1}. \qquad \square$$

What to take away. Dividing a binomial coefficient by $k+1$ corresponds to integrating the generating polynomial from 0 to 1. The same idea, integrating $x \cdot (1+x)^n$ or $x^2(1+x)^n$ instead, produces sums weighted by $1/((k+1)(k+2))$, and so on. Calculus on $(1+x)^n$ is the universal tool for extracting sums of the form $\sum w(k)\binom{n}{k}$ whenever the weight $w(k)$ is a polynomial in k or a rational function of the form $1/((k+a)(k+b)\cdots)$.

CONVOLUTION IDENTITIES

PROBLEM 13.5. Prove that for every non-negative integer n,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Technique. Compare the coefficient of x^n in both sides of the identity $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$.

Discussion. The left-hand side $(1+x)^{2n}$ has coefficient $\binom{2n}{n}$ on x^n.

On the right-hand side, expand both factors by the binomial theorem:

$$(1+x)^n \cdot (1+x)^n = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right).$$

The coefficient of x^n in this product is

$$\sum_{i+j=n} \binom{n}{i}\binom{n}{j} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}.$$

Using the symmetry $\binom{n}{n-i} = \binom{n}{i}$, this becomes $\sum_{i=0}^{n} \binom{n}{i}^2$.

Equating the two expressions for the coefficient of x^n,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \qquad \square$$

What to take away. The product $(1+x)^m(1+x)^n = (1+x)^{m+n}$ is a polynomial identity; expanding both sides and equating coefficients of x^r produces Vandermonde's identity in its full generality. The square-sum above is the special case $m = n = r$, and $\binom{2n}{n}$ is called the *central* binomial coefficient. Vandermonde's identity is therefore a single-line consequence of the multiplicative factorisation $(1+x)^{m+n} = (1+x)^m(1+x)^n$; every convolution identity for binomial coefficients has a similar generating-function origin.

Problem 13.6. Let r and n be non-negative integers with $r \leq n$. Prove the identity

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Technique. Rewrite each summand as a difference using Pascal's rule, and telescope.

Discussion. Pascal's rule, $\binom{k+1}{r+1} = \binom{k}{r} + \binom{k}{r+1}$, rearranges to

$$\binom{k}{r} = \binom{k+1}{r+1} - \binom{k}{r+1}.$$

Summing from $k = r$ to $k = n$, the right-hand side telescopes:

$$\sum_{k=r}^{n} \binom{k}{r} = \sum_{k=r}^{n} \left[\binom{k+1}{r+1} - \binom{k}{r+1} \right] = \binom{n+1}{r+1} - \binom{r}{r+1}.$$

The last term $\binom{r}{r+1} = 0$, so

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}. \qquad \square$$

What to take away. The identity gets its popular name from its appearance in Pascal's triangle: the sum of binomial coefficients along a diagonal running "down and to the right" starting from $\binom{r}{r}$ is the coefficient just below the diagonal's endpoint, and the path on the triangle resembles a hockey stick. Any identity of shape $\sum f(k) = F(n)$ with F compatibly defined can be attacked by writing $f(k)$ as a finite difference $F(k) - F(k-1)$ or, as here, $g(k+1) - g(k)$; the telescoping does the rest.

Coefficient extraction

Problem 13.7. Find the coefficient of x^8 in the expansion of $(1 + x + x^2 + \cdots + x^{10})^6$.

Technique. Rewrite the base polynomial as a finite geometric sum; extract the target coefficient using a generalised binomial series, with the upper-bound correction dropping out for degree reasons.

Discussion. The base is a finite geometric sum:

$$1 + x + x^2 + \cdots + x^{10} \;=\; \frac{1 - x^{11}}{1 - x}, \qquad x \neq 1.$$

Raising to the sixth power,

$$(1 + x + \cdots + x^{10})^6 \;=\; (1 - x^{11})^6 \cdot (1 - x)^{-6}.$$

In the expansion of $(1 - x^{11})^6 = 1 - 6x^{11} + \binom{6}{2}x^{22} - \cdots$ every non-constant term has degree at least 11. Since we want the coefficient of x^8, and $8 < 11$, only the constant term 1 from $(1 - x^{11})^6$ contributes. Therefore

$$[x^8](1 + x + \cdots + x^{10})^6 \;=\; [x^8](1 - x)^{-6}.$$

The generalised binomial series. For every positive integer n,

$$(1 - x)^{-n} \;=\; \sum_{k=0}^{\infty} \binom{n + k - 1}{k} x^k.$$

To see this, observe that $(1 - x)^{-n} = (1 + x + x^2 + \cdots)^n$, and the coefficient of x^k in this infinite product is the number of ways to write k as an ordered sum $a_1 + a_2 + \cdots + a_n$ with each $a_i \geq 0$. Represent such a choice by k stars and $n - 1$ bars in a row (the i-th group of stars supplies a_i); the number of such strings is the number of ways to choose the positions of the $n - 1$ bars from $n + k - 1$ total positions, namely $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$.

Applying this identity with $n = 6$ and $k = 8$,

$$[x^8](1 - x)^{-6} \;=\; \binom{13}{8} \;=\; \binom{13}{5} \;=\; 1287.$$

Hence

$$[x^8](1 + x + x^2 + \cdots + x^{10})^6 \;=\; 1287. \qquad \square$$

What to take away. The identity $1 + x + \cdots + x^m = (1 - x^{m+1})/(1 - x)$ converts any finite-alphabet generating function into a ratio of two polynomials. Paired with the generalised binomial series $(1 - x)^{-n} = \sum \binom{n+k-1}{k} x^k$, it turns "how many ways to choose n non-negative integers summing to k, each at most m" into a three-line coefficient extraction. Here the bound x^{10} was irrelevant because the target degree 8 was smaller than the smallest correction x^{11}; for a target exceeding 11, the $(1-x^{11})^6$ expansion contributes extra signed terms that must be tracked.

Problem 13.8. Find the coefficient of x^{15} in the expansion of $(1-3x+3x^2-x^3)^{17}$.

Technique. Recognise the inner polynomial as $(1-x)^3$; the expansion reduces to a single binomial.

Discussion. The inner polynomial is a perfect cube:

$$1-3x+3x^2-x^3 \;=\; (1-x)^3.$$

Therefore

$$(1-3x+3x^2-x^3)^{17} \;=\; (1-x)^{51}.$$

The coefficient of x^{15} in $(1-x)^{51}$ is

$$[x^{15}](1-x)^{51} \;=\; (-1)^{15}\binom{51}{15} \;=\; -\binom{51}{15}.$$

□

What to take away. Before mechanically writing down the general term of a large product, look for a factorisation. Here the seventeenth power of a cube is the fifty-first power of a binomial, and the coefficient falls out with no summation at all. Perfect-cube recognition $(1\pm 3x+3x^2\pm x^3=(1\pm x)^3)$ and perfect-square recognition $(1\pm 2x+x^2=(1\pm x)^2)$ are the two factorisations to keep in muscle memory.

Monotonicity and roots of unity

Problem 13.9. Prove that for every positive integer n, the central coefficient $\binom{2n}{n}$ is strictly greater than every other coefficient $\binom{2n}{k}$ with $k \neq n$.

Technique. Study the ratio of consecutive coefficients and show it crosses 1 at the centre.

Discussion. For $0 \le k \le 2n-1$, compute the ratio

$$\frac{\binom{2n}{k+1}}{\binom{2n}{k}} = \frac{(2n)!/[(k+1)!(2n-k-1)!]}{(2n)!/[k!(2n-k)!]} = \frac{2n-k}{k+1}.$$

The ratio exceeds 1 precisely when $2n-k > k+1$, that is, when $k < n - \frac{1}{2}$, equivalently $k \le n-1$.

Increasing phase. For $k = 0, 1, \dots, n-1$, the ratio is greater than 1, so

$$\binom{2n}{0} < \binom{2n}{1} < \cdots < \binom{2n}{n}.$$

Decreasing phase. For $k = n, n+1, \dots, 2n-1$, the ratio $(2n-k)/(k+1)$ is less than 1 (at $k = n$ the ratio equals $n/(n+1) < 1$), so

$$\binom{2n}{n} > \binom{2n}{n+1} > \cdots > \binom{2n}{2n}.$$

Combining the two phases, $\binom{2n}{n}$ is strictly greater than every other coefficient in the $2n$-th row of Pascal's triangle. □

What to take away. The ratio $(2n-k)/(k+1)$ is the engine: it encodes both how coefficients grow on the way up and how they shrink on the way down. For rows of odd index $2n+1$ the analogous argument shows that there are two tied maxima, $\binom{2n+1}{n} = \binom{2n+1}{n+1}$. In both cases, the maximum value is exponentially large in n: for even rows, $\binom{2n}{n} \sim 4^n/\sqrt{\pi n}$ by Stirling's approximation. It is the single largest term in the row, but it does not dominate the row's total: the whole row sums to 4^n, so the central coefficient's share is $\binom{2n}{n}/4^n \sim 1/\sqrt{\pi n}$, which tends to 0. The peak stays exponentially tall while the mass spreads ever wider around it.

Problem 13.10. For every positive integer n, prove that

$$\sum_{\substack{0 \le k \le n \\ k \equiv 0 \pmod 3}} \binom{n}{k} = \frac{1}{3}(2^n + 2\cos(n\pi/3)).$$

Technique. Evaluate $(1+x)^n$ at $x = 1$, ω, ω^2, where $\omega = e^{2\pi i/3}$ is a primitive cube root of unity; average the three values to filter the terms with $k \equiv 0 \pmod 3$.

Discussion. Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$. Then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. For any integer k, the sum $1 + \omega^k + \omega^{2k}$ depends only on k mod 3:

- if $k \equiv 0 \pmod 3$, then $\omega^k = \omega^{2k} = 1$ and the sum is 3;
- otherwise, ω^k and ω^{2k} are the two non-trivial cube roots of unity and the sum is 0.

Apply the binomial theorem at $x = 1, \omega, \omega^2$ and add:

$$(1+1)^n + (1+\omega)^n + (1+\omega^2)^n \;=\; \sum_{k=0}^{n} \binom{n}{k}(1 + \omega^k + \omega^{2k}).$$

The right-hand side equals $3\sum_{k\equiv 0 \pmod 3}\binom{n}{k}$ by the case analysis above.

Now simplify the left-hand side. The first term is 2^n. For the other two, compute $1+\omega$ and $1+\omega^2$ in polar form. Using $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$,

$$1+\omega \;=\; \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}\mathrm{i} \;=\; \mathrm{e}^{\mathrm{i}\pi/3}, \qquad 1+\omega^2 \;=\; \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\mathrm{i} \;=\; \mathrm{e}^{-\mathrm{i}\pi/3}.$$

Hence $(1+\omega)^n + (1+\omega^2)^n = \mathrm{e}^{\mathrm{i}n\pi/3} + \mathrm{e}^{-\mathrm{i}n\pi/3} = 2\cos(n\pi/3)$, and the left-hand side is $2^n + 2\cos(n\pi/3)$.

Dividing by 3,

$$\sum_{k\equiv 0 \pmod 3} \binom{n}{k} \;=\; \frac{1}{3}(2^n + 2\cos(n\pi/3)). \qquad \square$$

What to take away. The roots-of-unity filter extracts any residue class of binomial coefficients. For a filter modulo m, evaluate $(1+x)^n$ at each m-th root of unity $1, \zeta, \zeta^2, \ldots, \zeta^{m-1}$ and average. The argument generalises any "generating function $f(x)$" to "sum of coefficients at indices $\equiv r \pmod m$":

$$\sum_{k\equiv r \pmod m} [x^k]f(x) \;=\; \frac{1}{m}\sum_{j=0}^{m-1} \zeta^{-jr} f(\zeta^j).$$

Every similar residue-class sum, no matter how intricate-looking, reduces to evaluating the generating polynomial at m-th roots of unity and averaging the results.

Chapter 14

Classical inequalities

✠

Two fundamental inequalities

Problem 14.1. Prove that for every integer $n \geq 2$ and every tuple of positive reals $x_1, x_2, \dots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n},$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Remark. This inequality is also proved in Chapter 6 (Problem 6.7), where it serves as the showcase for Cauchy's forward-backward induction: there the induction is the lesson. Here AM-GM is the foundation the rest of the chapter rests on, used again and again below, so we restate it and recall its proof to keep the chapter self-contained. The two appearances serve different ends, the technique there and the tool here.

Technique. Cauchy's forward-backward induction: prove $n = 2$; show that the inequality at n implies the inequality at $2n$ (forward); show that the inequality at $n + 1$ implies the inequality at n (backward). The two implications together cover every integer $n \geq 2$.

Discussion. The case $n = 2$. For positive a, b,

$$\left(\frac{a+b}{2}\right)^2 - ab = \frac{(a-b)^2}{4} \geq 0,$$

with equality iff $a = b$. Taking square roots gives $\frac{a+b}{2} \geq \sqrt{ab}$.

Forward step: AM-GM at n implies AM-GM at $2n$. Let $y_1, \dots, y_{2n}$ be positive. Split into two halves of size n. By the hypothesis applied to each half,

$$\frac{y_1 + \cdots + y_n}{n} \;\geq\; \sqrt[n]{y_1 \cdots y_n},$$

$$\frac{y_{n+1} + \cdots + y_{2n}}{n} \;\geq\; \sqrt[n]{y_{n+1} \cdots y_{2n}}.$$

Set $P = \sqrt[n]{y_1 \cdots y_n}$ and $Q = \sqrt[n]{y_{n+1} \cdots y_{2n}}$. Apply the $n = 2$ case to P and Q:

$$\frac{P+Q}{2} \;\geq\; \sqrt{PQ} \;=\; \sqrt[2n]{y_1 \cdots y_{2n}}.$$

The average of the two n-term arithmetic means is $(y_1 + \cdots + y_{2n})/(2n)$, so chaining gives AM-GM at $2n$.

Backward step: AM-GM at $n+1$ implies AM-GM at n. Let $x_1, \dots, x_n$ be positive, and write $A = (x_1 + \cdots + x_n)/n$. Apply AM-GM at $n+1$ to the tuple $(x_1, \dots, x_n, A)$:

$$\frac{x_1 + \cdots + x_n + A}{n+1} \;\geq\; \sqrt[n+1]{x_1 \cdots x_n \cdot A}.$$

The left side equals A, so $A \geq \sqrt[n+1]{x_1 \cdots x_n \cdot A}$. Raising both sides to the power $n+1$ and dividing by A,

$$A^{n+1} \;\geq\; x_1 \cdots x_n \cdot A, \qquad \text{hence} \qquad A^n \;\geq\; x_1 \cdots x_n,$$

which is AM-GM at n.

Combining. The forward step climbs by doubling: from $n = 2$ to $n = 4, 8, 16, \ldots$, all powers of 2. The backward step descends one at a time. Every integer $n \geq 2$ is reached: pick the smallest power of 2 that exceeds n, descend back down to n. Equality propagates through each step and holds only when all arguments coincide. □

What to take away. AM-GM is the most reused inequality in classical algebra. The forward-backward technique is itself worth retaining: when ordinary one-step induction fails, double-then-descend often succeeds. The crucial trick in the backward step is to use the arithmetic mean A as the auxiliary entry. This collapses the larger inequality down to the smaller one in a single algebraic move.

Problem 14.2. Prove that for any real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right),$$

with equality if and only if the sequences (a_i) and (b_i) are proportional.

Technique. The discriminant trick: a quadratic in the parameter t that is non-negative for every real t must have non-positive discriminant.

Discussion. Consider the function of t:

$$Q(t) = \sum_{i=1}^{n} (a_i t - b_i)^2.$$

As a sum of squares, $Q(t) \geq 0$ for every real t. Expand:

$$Q(t) = t^2 \sum a_i^2 - 2t \sum a_i b_i + \sum b_i^2.$$

This is a quadratic in t with leading coefficient $\sum a_i^2 \geq 0$. Assuming $\sum a_i^2 > 0$ (otherwise all a_i are zero and the inequality is the trivial $0 \leq 0$), Q is an upward-opening parabola that never dips below the t-axis. Its discriminant must therefore be non-positive:

$$(-2 \sum a_i b_i)^2 - 4(\sum a_i^2)(\sum b_i^2) \leq 0.$$

Dividing by 4 and rearranging,

$$\left(\sum a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

Equality. Suppose first that not all a_i are zero, so $Q(t) = \sum (a_i t - b_i)^2$ is a genuine upward quadratic. Equality holds iff Q has a real root t_0, which happens iff every term $(a_i t_0 - b_i)^2$ vanishes, that is, $b_i = t_0 a_i$ for every i: the sequences are proportional. The degenerate case $a_i = 0$ for all i makes both sides 0, so equality holds for every (b_i); this is consistent with "proportional" if one allows the zero sequence as proportional to anything. □

What to take away. The Cauchy-Schwarz inequality is the second pillar of classical inequalities, alongside AM-GM. Its discriminant proof is one line once the auxiliary $Q(t)$ is in place. The same technique evaluates a remarkable variety of quadratic-shaped inequalities in any number of variables: write down the right "sum of squares is non-negative" identity, and the inequality reads off as the discriminant condition.

APPLICATIONS OF AM-GM

PROBLEM 14.3. For positive reals a, b, c, prove that

$$(a+b)(b+c)(c+a) \;\geq\; 8abc,$$

with equality if and only if $a = b = c$.

Technique. Apply the two-variable AM-GM separately to each factor on the left, and multiply.

Discussion. By the case $n = 2$ of Problem 14.1,

$$a+b \;\geq\; 2\sqrt{ab}, \qquad b+c \;\geq\; 2\sqrt{bc}, \qquad c+a \;\geq\; 2\sqrt{ca}.$$

Multiplying the three inequalities (all sides positive),

$$(a+b)(b+c)(c+a) \;\geq\; 8\sqrt{ab\cdot bc\cdot ca} \;=\; 8\sqrt{(abc)^2} \;=\; 8abc.$$

Equality requires equality in each two-variable AM-GM, that is, $a = b$, $b = c$, $c = a$. The three conditions together force $a = b = c$. □

What to take away. A symmetric inequality of the form "product of pairwise sums beats product" frequently dissolves into a chain of two-variable AM-GMs, multiplied. The strategy generalises: $(x_1 + x_2)(x_2 + x_3)\cdots(x_n + x_1) \geq 2^n\, x_1x_2\cdots x_n$ for positive reals, with the same proof.

PROBLEM 14.4. For positive reals a, b, c with $a+b+c = 1$, prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \;\geq\; 64.$$

Technique. Use the constraint to rewrite each factor; apply the four-variable AM-GM to the resulting numerator; multiply.

Discussion. Substitute $1 = a + b + c$ in the numerator of $1 + 1/a = (a + 1)/a$:

$$1 + \frac{1}{a} = \frac{a + (a + b + c)}{a} = \frac{2a + b + c}{a} = \frac{a + a + b + c}{a}.$$

Apply AM-GM (Problem 14.1) to the four positives a, a, b, c:

$$\frac{a + a + b + c}{4} \geq \sqrt[4]{a \cdot a \cdot b \cdot c} = \sqrt[4]{a^2bc}.$$

Therefore

$$1 + \frac{1}{a} \geq \frac{4\sqrt[4]{a^2bc}}{a} = 4\sqrt[4]{\frac{bc}{a^2}}.$$

By symmetry, $1 + 1/b \geq 4\sqrt[4]{ca/b^2}$ and $1 + 1/c \geq 4\sqrt[4]{ab/c^2}$. Multiplying the three inequalities,

$$\prod_{\text{cyc}} \left(1 + \frac{1}{a}\right) \geq 64\sqrt[4]{\frac{bc}{a^2} \cdot \frac{ca}{b^2} \cdot \frac{ab}{c^2}} = 64\sqrt[4]{\frac{(abc)^2}{(abc)^2}} = 64.$$

Equality in each step requires $a = a = b = c$, i.e., $a = b = c$, combined with $a + b + c = 1$, gives $a = b = c = \frac{1}{3}$. At that point each factor equals 4 and the product equals 64. □

What to take away. A constraint of the form "the sum of variables equals a fixed total" is a free pass to rewrite every 1 in the expression as a sum. Once $1 + 1/a$ becomes $(a + a + b + c)/a$, AM-GM on the four-term numerator does the rest. This trick ("replace constants by the constraint") is standard in olympiad-style optimisation problems.

APPLICATIONS OF CAUCHY-SCHWARZ

PROBLEM 14.5. For real numbers $a_1, \dots, a_n$ and positive reals $b_1, \dots, b_n$, prove the *Engel form* of Cauchy-Schwarz:

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i},$$

with equality if and only if the ratio a_i/b_i is constant in i.

Technique. Apply Cauchy-Schwarz to the rescaled sequences $(a_i/\sqrt{b_i})$ and $(\sqrt{b_i})$.

Discussion. By Cauchy-Schwarz (Problem 14.2) applied with $u_i = a_i/\sqrt{b_i}$ and $v_i = \sqrt{b_i}$,

$$\left(\sum u_i v_i\right)^2 \leq \left(\sum u_i^2\right)\left(\sum v_i^2\right).$$

Substituting,

$$\left(\sum a_i\right)^2 \leq \left(\sum \frac{a_i^2}{b_i}\right)\left(\sum b_i\right).$$

Dividing by the positive quantity $\sum b_i$ gives the claimed inequality. Equality holds iff u_i and v_i are proportional, i.e. $a_i/\sqrt{b_i} = \lambda\sqrt{b_i}$, i.e. a_i/b_i is constant. □

What to take away. The Engel form is the workhorse for proving inequalities involving sums of squares-over-positives. Its strength: the right-hand side has a single denominator $\sum b_i$, which often telescopes or simplifies in practice (as in the next problem). Whenever a problem features $\sum a_i^2/b_i$, the Engel form is the first tool to try.

PROBLEM 14.6. For positive reals a, b, c, prove the *Nesbitt inequality*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2},$$

with equality if and only if $a = b = c$.

Technique. Apply the Engel form (Problem 14.5); reduce to a well-known three-variable identity.

Discussion. Multiply each numerator and denominator by the corresponding numerator: $a/(b+c) = a^2/[a(b+c)]$, and similarly for the other two terms. The sum becomes

$$\sum_{\text{cyc}} \frac{a^2}{a(b+c)}.$$

Apply Engel form with $a_i \in \{a, b, c\}$ and $b_i \in \{a(b+c), b(c+a), c(a+b)\}$:

$$\sum_{\text{cyc}} \frac{a^2}{a(b+c)} \;\geq\; \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)}.$$

The denominator on the right expands to $2(ab+bc+ca)$, so

$$\sum_{\text{cyc}} \frac{a}{b+c} \;\geq\; \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

It remains to show $(a+b+c)^2 \geq 3(ab+bc+ca)$. Expanding gives

$$(a+b+c)^2 - 3(ab+bc+ca) \;=\; a^2+b^2+c^2-ab-bc-ca,$$

which equals $\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$, a non-negative quantity that vanishes iff $a=b=c$. Therefore

$$\sum_{\text{cyc}} \frac{a}{b+c} \;\geq\; \frac{3(ab+bc+ca)}{2(ab+bc+ca)} \;=\; \frac{3}{2},$$

with equality iff $a=b=c$. □

What to take away. Nesbitt is the prototype of a "cyclic-sum" inequality. The two-step proof (Engel, then a small symmetric identity) is the standard route. The auxiliary identity $(a+b+c)^2 \geq 3(ab+bc+ca)$ is itself a frequent visitor and worth remembering: it is the three-variable analogue of $a^2+b^2 \geq 2ab$.

POWER MEANS AND REARRANGEMENT

PROBLEM 14.7. For positive reals $x_1, \dots, x_n$, define the harmonic, geometric, arithmetic, and quadratic means by

$$\mathrm{HM} = \frac{n}{\sum 1/x_i}, \quad \mathrm{GM} = \sqrt[n]{x_1 \cdots x_n},$$

$$\mathrm{AM} = \frac{\sum x_i}{n}, \quad \mathrm{QM} = \sqrt{\frac{\sum x_i^2}{n}}.$$

Prove the chain $\mathrm{HM} \le \mathrm{GM} \le \mathrm{AM} \le \mathrm{QM}$, with equality throughout if and only if all x_i are equal.

Technique. Each of the three inequalities reduces to one of the two fundamental inequalities of this chapter, applied to the right rearrangement of the data.

Discussion. GM ≤ AM. This is exactly Problem 14.1, applied to $(x_1, \dots, x_n)$.

HM ≤ GM. Apply AM-GM to the reciprocals $1/x_1, \dots, 1/x_n$:

$$\frac{\sum 1/x_i}{n} \ge \sqrt[n]{\prod \frac{1}{x_i}} = \frac{1}{\sqrt[n]{\prod x_i}} = \frac{1}{\mathrm{GM}}.$$

Take reciprocals (which reverses the inequality, both sides positive):

$$\frac{n}{\sum 1/x_i} \le \mathrm{GM}, \qquad \text{that is,} \qquad \mathrm{HM} \le \mathrm{GM}.$$

AM ≤ QM. Apply Cauchy-Schwarz (Problem 14.2) to $(x_1, \dots, x_n)$ and $(1, 1, \dots, 1)$:

$$\left(\sum x_i \cdot 1\right)^2 \le \left(\sum x_i^2\right)\left(\sum 1^2\right) = n \sum x_i^2.$$

Divide both sides by n^2:

$$\left(\frac{\sum x_i}{n}\right)^2 \le \frac{\sum x_i^2}{n}, \qquad \text{that is,} \qquad \mathrm{AM}^2 \le \mathrm{QM}^2.$$

Both means are positive, so AM ≤ QM.

Each step is an equality iff all x_i are equal (the equality cases of the two fundamental inequalities). Combining, equality throughout the chain forces $x_1 = \cdots = x_n$. □

What to take away. The four classical means sit in a single linear order, controlled by AM-GM and Cauchy-Schwarz. The chain extends to a continuous family: for any real r, the *power mean* $M_r = (\frac{1}{n}\sum x_i^r)^{1/r}$ is a monotone function of r on positive reals, with $M_{-1} = \mathrm{HM}$, $M_0 = \mathrm{GM}$ (in the limit), $M_1 = \mathrm{AM}$, $M_2 = \mathrm{QM}$. The general monotonicity $M_r \leq M_s$ for $r < s$ is a standard theorem, though not a one-line one: it follows from Jensen's inequality applied to a suitable power function, with the direction of convexity (and so of the inequality) depending on the signs of r and s. One cannot simply invoke "x^r is convex," since x^r is in fact concave for $0 < r < 1$.

Problem 14.8. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be two non-decreasing sequences of real numbers. Prove *Chebyshev's sum inequality*:

$$n\sum_{i=1}^{n} a_i b_i \;\geq\; \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

Technique. Sum the inequality $(a_i - a_j)(b_i - b_j) \geq 0$ over all pairs (i, j), and rearrange.

Discussion. For any indices i, j in $\{1, \dots, n\}$, the differences $a_i - a_j$ and $b_i - b_j$ have the same sign (both non-negative if $i \geq j$, both non-positive if $i \leq j$), because both sequences are non-decreasing. Therefore

$$(a_i - a_j)(b_i - b_j) \;\geq\; 0 \qquad \text{for every } i, j.$$

Sum over all n^2 ordered pairs:

$$\sum_{i,j}(a_i - a_j)(b_i - b_j) \;\geq\; 0.$$

Expand the left side:

$$\sum_{i,j}(a_ib_i + a_jb_j - a_ib_j - a_jb_i) \;=\; 2n\sum_i a_ib_i \;-\; 2\left(\sum_i a_i\right)\left(\sum_j b_j\right).$$

(The first two summands each contribute $n\sum_i a_ib_i$; the last two each contribute $(\sum a_i)(\sum b_j)$.) Dividing by 2 and rearranging,

$$n\sum_i a_ib_i \;\geq\; \left(\sum a_i\right)\left(\sum b_i\right),$$

with equality iff every $(a_i - a_j)(b_i - b_j) = 0$, i.e., one of the two sequences is constant. □

What to take away. Chebyshev's inequality says that pairing same-sorted sequences gives a sum that beats the average pairing. It is the symmetric companion to the rearrangement inequality, which says: for fixed multisets $\{a_i\}$ and $\{b_i\}$, the sum $\sum a_i b_{\sigma(i)}$ is maximised when both sequences are sorted in the same order, and minimised when they are sorted oppositely. The same "sum of $(a_i - a_j)(b_i - b_j) \geq 0$" trick proves both.

HARDER APPLICATIONS

PROBLEM 14.9. Let a, b, c be the side lengths of a triangle with semi-perimeter $s = (a + b + c)/2$. Prove that

$$(s-a)(s-b)(s-c) \;\leq\; \frac{s^3}{27},$$

with equality if and only if $a = b = c$. Conclude (via Heron's formula) that among all triangles of a given perimeter, the equilateral one has the largest area.

Technique. The three positives $s-a, s-b, s-c$ sum to s. Apply AM-GM to them.

Discussion. By the triangle inequality, $a + b > c$, $b + c > a$, $c + a > b$, so each of $s - a$, $s - b$, $s - c$ is strictly positive. Their sum is

$$(s-a) + (s-b) + (s-c) \;=\; 3s - (a+b+c) \;=\; 3s - 2s \;=\; s.$$

Apply AM-GM (Problem 14.1) to the three positives:

$$\sqrt[3]{(s-a)(s-b)(s-c)} \;\leq\; \frac{(s-a)+(s-b)+(s-c)}{3} \;=\; \frac{s}{3}.$$

Cubing both sides,

$$(s-a)(s-b)(s-c) \;\leq\; \frac{s^3}{27},$$

with equality iff $s-a=s-b=s-c$, i.e., $a=b=c$.

Conclusion via Heron's formula. Heron's formula expresses the area of a triangle with sides a, b, c and semi-perimeter s as

$$\text{Area} \;=\; \sqrt{s(s-a)(s-b)(s-c)}.$$

Applying the inequality just proved,

$$\text{Area} \;\le\; \sqrt{s\cdot\frac{s^3}{27}} \;=\; \frac{s^2}{\sqrt{27}} \;=\; \frac{s^2}{3\sqrt{3}}.$$

For fixed perimeter $2s$, the right-hand side is a constant, and the bound is attained iff $a=b=c=2s/3$, the equilateral triangle. So among triangles of fixed perimeter, the equilateral maximises area. □

What to take away. A classical isoperimetric-style result (n-gons of fixed perimeter, the regular one has maximum area) reduces, in the triangle case, to a one-line application of AM-GM on three positives whose sum is the semi-perimeter. Heron's formula transforms the area question into a product question, and AM-GM bounds the product. This is the model for many geometric extremum problems: convert area or volume into an algebraic product, then apply AM-GM.

Problem 14.10. For positive reals a, b, c satisfying $abc = 1$, prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \;\ge\; \frac{3}{2}.$$

Technique. Substitute $a=1/x$, $b=1/y$, $c=1/z$ with $xyz=1$ to convert each term into the Engel-form template $x^2/(y+z)$; apply Engel form (Problem 14.5); finish with AM-GM.

Discussion. The substitution. Set $a=1/x$, $b=1/y$, $c=1/z$, where x, y, z are positive reals. The constraint $abc=1$ becomes $1/(xyz)=1$, i.e. $xyz=1$.

Compute one term of the sum:

$$a^3(b+c) \;=\; \frac{1}{x^3}\left(\frac{1}{y}+\frac{1}{z}\right) \;=\; \frac{y+z}{x^3yz}.$$

Using $xyz = 1$, we have $yz = 1/x$, so $a^3(b+c) = (y+z)/x^2$. Inverting,

$$\frac{1}{a^3(b+c)} = \frac{x^2}{y+z}.$$

By symmetry, the inequality becomes

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2} \qquad \text{for } xyz = 1.$$

Engel form. By the Engel-form inequality (Problem 14.5) applied to numerators $\{x, y, z\}$ and denominators $\{y+z, z+x, x+y\}$,

$$\sum_{\text{cyc}} \frac{x^2}{y+z} \geq \frac{(x+y+z)^2}{(y+z)+(z+x)+(x+y)} = \frac{x+y+z}{2},$$

since the denominator on the right simplifies to $2(x+y+z)$.

Finish via AM-GM. By AM-GM (Problem 14.1) on three positives,

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = \sqrt[3]{1} = 1, \qquad \text{hence} \qquad x+y+z \geq 3.$$

Combining the two,

$$\sum_{\text{cyc}} \frac{x^2}{y+z} \geq \frac{x+y+z}{2} \geq \frac{3}{2},$$

which is the desired inequality. Equality holds iff equality holds in both Engel form ($x = y = z$) and AM-GM ($x = y = z$), and these are consistent with $xyz = 1$ only when $x = y = z = 1$, i.e. $a = b = c = 1$. □

What to take away. The chapter culminates in this problem because it brings together every tool: a clever substitution to reshape the expression, the Engel form to bound the rearranged sum below, and the fundamental AM-GM to close the gap. The substitution $a = 1/x$ is the trick that often unlocks symmetric inequalities under the constraint $abc = 1$: it swaps factors of a^k for factors of $1/x^k$ and converts "cubic in the denominator" problems into "squared in the numerator" problems. With the right substitution, even an IMO problem reduces to two named inequalities applied in succession.

CHAPTER 15

Functional equations

SUBSTITUTION SYSTEMS

PROBLEM 15.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + 2f(1-x) \;=\; x^2 \qquad \text{for every real } x.$$

Technique. The substitution $x \mapsto 1-x$ is an involution, applying it produces a second equation, and the resulting 2×2 linear system solves for $f(x)$.

Discussion. Call the given identity equation (i):

$$f(x) + 2f(1-x) \;=\; x^2. \tag{i}$$

Substitute $x \mapsto 1-x$. The left-hand side becomes $f(1-x) + 2f(x)$, and the right-hand side becomes $(1-x)^2$, so

$$2f(x) + f(1-x) \;=\; (1-x)^2 \;=\; x^2 - 2x + 1. \tag{ii}$$

Treat (i) and (ii) as a linear system in the two unknowns $f(x)$ and $f(1-x)$. Multiply (ii) by 2 and subtract (i):

$$3f(x) \;=\; 2(x^2 - 2x + 1) - x^2 \;=\; x^2 - 4x + 2.$$

Therefore

$$f(x) \;=\; \frac{x^2 - 4x + 2}{3}.$$

A check: $f(1-x) = (x^2 + 2x - 1)/3$, so

$$f(x) + 2f(1-x) \;=\; \frac{(x^2 - 4x + 2) + 2(x^2 + 2x - 1)}{3} \;=\; x^2,$$

as required. □

What to take away. Whenever the same two unknowns $f(x)$ and $f(T(x))$ appear in a functional equation with T an involution ($T \circ T$ = identity), applying T produces a second equation, and the pair together is a 2×2 linear system. Solving is mechanical once the system is in place; the hard part is recognising the involutive substitution.

Problem 15.2. Find all functions $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ satisfying

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x \qquad \text{for every } x \neq 0, 1.$$

Technique. The substitution $T(x) = 1 - 1/x$ has order three (three applications return to x), so the equation together with its two iterates forms a 3×3 linear system in $f(x), f(Tx), f(T^2x)$.

Discussion. Compute the iterates of T:

$$T(x) = 1 - \frac{1}{x}, \qquad T^2(x) = \frac{1}{1-x}, \qquad T^3(x) = x.$$

So T has order three on $\mathbb{R} \setminus \{0,1\}$.

Apply the given identity at x, at $T(x)$, and at $T^2(x)$:

$$f(x) + f(Tx) = 1 + x, \tag{i}$$

$$f(Tx) + f(T^2x) = 1 + Tx = 2 - \frac{1}{x}, \tag{ii}$$

$$f(T^2x) + f(x) = 1 + T^2x = \frac{2-x}{1-x}. \tag{iii}$$

Adding (i) and (iii) and subtracting (ii),

$$2f(x) = (1+x) + \frac{2-x}{1-x} - \left(2 - \frac{1}{x}\right).$$

Simplify $(2-x)/(1-x) = 1 + 1/(1-x)$:

$$2f(x) = (1+x) + 1 + \frac{1}{1-x} - 2 + \frac{1}{x} = x + \frac{1}{x} + \frac{1}{1-x}.$$

Therefore

$$f(x) = \frac{1}{2}\left(x + \frac{1}{x} + \frac{1}{1-x}\right).$$

A check: with $Tx = (x-1)/x$, computing $f(Tx)$ and adding gives $f(x) + f(Tx) = 1 + x$, as required. □

What to take away. The order of the substitution T is the size of the linear system the equation generates. Order-two substitutions yield 2×2 systems; order-three substitutions yield 3×3 systems. For order k, the system is $k \times k$, and it determines $f(x)$ uniquely whenever its coefficient matrix is non-singular, as it is in the cases here. That is not automatic: for particular equations the matrix can be singular, in which case the system either fails to pin down $f(x)$ or forces a consistency condition on the constants. When you meet a functional equation, the first diagnostic is: what is the order of the substitution implicit in it?

CAUCHY'S ADDITIVE EQUATION

PROBLEM 15.3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying *Cauchy's functional equation*

$$f(x+y) = f(x) + f(y) \qquad \text{for every } x, y \in \mathbb{R}.$$

Technique. Determine f on $\mathbb{Z}$ by induction, extend to $\mathbb{Q}$ via $qf(p/q) = f(p)$, and extend to $\mathbb{R}$ by continuity plus density of $\mathbb{Q}$.

Discussion. Set $x = y = 0$: $f(0) = 2f(0)$, so $f(0) = 0$. Set $y = -x$: $f(0) = f(x) + f(-x)$, so $f(-x) = -f(x)$: f is odd.

Values on positive integers. By induction on n: the base case $n = 1$ is trivial, and if $f(n) = nf(1)$, then $f(n+1) = f(n) + f(1) = (n+1)f(1)$. By odd symmetry, $f(n) = nf(1)$ for every integer n.

Values on rationals. For positive integers p, q: iterating the equation q times yields $f(q \cdot p/q) = qf(p/q)$. But the left-hand side is $f(p) = pf(1)$. Therefore

$$f(p/q) = \frac{p}{q} f(1).$$

Combined with odd symmetry, $f(r) = rf(1)$ for every $r \in \mathbb{Q}$.

Extension to reals. Every real x is the limit of a sequence $r_n \to x$ of rationals. By continuity of f,

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n f(1) = x f(1).$$

Setting $c = f(1)$: the continuous solutions are exactly $f(x) = cx$. □

What to take away. Cauchy's equation determines f completely on rationals from additivity alone, and continuity then fixes f on all of $\mathbb{R}$. Without continuity the conclusion fails: using the axiom of choice one can construct pathological additive functions on $\mathbb{R}$ that are not linear ("Hamel-basis" functions), unbounded on every interval. The mildness of the regularity required, just continuity, is a recurring theme in functional equations: monotonicity, boundedness on an interval, or even measurability each suffice.

Problem 15.4. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) \;=\; f(x) + f(y) + xy \qquad \text{for every } x, y \in \mathbb{R}.$$

Technique. A suitable polynomial shift absorbs the xy term, reducing the equation to Cauchy's.

Discussion. Substitute $g(x) = f(x) - \frac{1}{2}x^2$. The given identity becomes

$$g(x+y) + \tfrac{1}{2}(x+y)^2 \;=\; g(x) + \tfrac{1}{2}x^2 + g(y) + \tfrac{1}{2}y^2 + xy.$$

The quadratic $\frac{1}{2}(x+y)^2 = \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy$ cancels the right-hand side's quadratic and cross term together, leaving

$$g(x+y) \;=\; g(x) + g(y).$$

So g is continuous and satisfies Cauchy's equation. By Problem 15.3, $g(x) = cx$. Therefore

$$f(x) \;=\; g(x) + \tfrac{1}{2}x^2 \;=\; \frac{x^2}{2} + cx.$$

A check: $f(x+y) = \frac{1}{2}(x+y)^2 + c(x+y)$ and $f(x) + f(y) + xy = \frac{x^2}{2} + cx + \frac{y^2}{2} + cy + xy$, which agree. □

What to take away. The extra term xy on the right is a red herring: it is the mixed term of a perfect square, and subtracting the ansatz $\frac{1}{2}x^2$ from f removes it entirely. The same shift trick works whenever the extra term matches the expansion of a specific polynomial in $x + y$. Problem-solving strategy: when a functional equation looks Cauchy-adjacent, try a polynomial ansatz for the discrepancy.

MULTIPLICATIVE AND EXPONENTIAL CAUCHY

PROBLEM 15.5. Find all continuous functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$f(xy) = f(x) + f(y) \qquad \text{for all } x, y > 0.$$

Technique. Compose f with the exponential to convert the multiplicative equation into Cauchy's additive equation.

Discussion. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) = f(\mathrm{e}^t)$. Continuity of e^t and of f makes g continuous. For every $s, t \in \mathbb{R}$,

$$g(s+t) = f(\mathrm{e}^{s+t}) = f(\mathrm{e}^s \mathrm{e}^t) = f(\mathrm{e}^s) + f(\mathrm{e}^t) = g(s) + g(t).$$

So g is continuous and satisfies Cauchy's additive equation. By Problem 15.3, $g(t) = ct$. Therefore

$$f(x) = g(\log x) = c \log x.$$

A check: $f(xy) = c\log(xy) = c(\log x + \log y) = f(x) + f(y)$, as required. □

What to take away. The exponential map converts multiplication into addition. Any continuous f on $(0, \infty)$ with $f(xy) = f(x) + f(y)$ is a continuous additive function of $\log x$, hence $f(x) = c\log x$. This is the functional-equation characterisation of the logarithm: every property of log that relies only on the additive law follows from this equation alone.

PROBLEM 15.6. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x)f(y) \qquad \text{for every } x, y \in \mathbb{R}.$$

Technique. Either f vanishes somewhere, in which case $f \equiv 0$; or f is everywhere positive, in which case $\log f$ is continuous additive and so linear.

Discussion. First observe $f(x) = f(x/2 + x/2) = f(x/2)^2 \geq 0$ for every x. So f is non-negative everywhere.

Case (i)*:* $f(x_0) = 0$ *for some* x_0. For any x, write $x = x_0 + (x - x_0)$:

$$f(x) \;=\; f(x_0)f(x - x_0) \;=\; 0.$$

So $f \equiv 0$.

Case (ii)*:* f *is nowhere zero.* Then $f(x) > 0$ for every x. Define $g(x) = \log f(x)$; g is continuous. From $f(x+y) = f(x)f(y)$,

$$g(x+y) \;=\; \log(f(x)f(y)) \;=\; g(x) + g(y).$$

By Problem 15.3, $g(x) = cx$, so $f(x) = \mathrm{e}^{cx}$. Setting $a = \mathrm{e}^{c} > 0$,

$$f(x) \;=\; a^x.$$

The continuous solutions are therefore $f \equiv 0$ and $f(x) = a^x$ for any $a > 0$. □

What to take away. Cauchy's *exponential* equation characterises the exponential function: every continuous solution is a^x (or identically zero). The proof proceeds by reducing to the additive Cauchy equation via the logarithm, after handling the zero case separately. The "one zero value forces identical zero" argument is a prototype for treating absorbing elements in a multiplicative functional equation.

MEAN-VALUE EQUATIONS

PROBLEM 15.7. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying *Jensen's functional equation*

$$f\Big(\frac{x+y}{2}\Big) \;=\; \frac{f(x) + f(y)}{2} \qquad \text{for every } x, y \in \mathbb{R}.$$

Technique. Subtract $f(0)$ to obtain a function g with $g(0) = 0$; derive Cauchy's additive equation for g; conclude that f is affine.

Discussion. Set $g(x) = f(x) - f(0)$. Substituting into Jensen's equation,

$$g\Big(\frac{x+y}{2}\Big) + f(0) \;=\; \frac{g(x) + f(0) + g(y) + f(0)}{2},$$

which simplifies to

$$g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}.$$

Setting $y = 0$ (and using $g(0) = 0$) gives $g(x/2) = g(x)/2$. Combining the midpoint identity for g with this half-scaling gives $g(x+y)/2 = (g(x)+g(y))/2$, that is,

$$g(x+y) = g(x)+g(y).$$

So g is continuous Cauchy-additive. By Problem 15.3, $g(x) = cx$, hence $f(x) = cx + f(0)$. Writing $a = c$ and $b = f(0)$, the continuous solutions are the affine functions

$$f(x) = ax+b. \qquad \square$$

What to take away. Jensen's equation is Cauchy's equation, plus a constant. The continuous solutions form a two-parameter family (slope a and intercept b), not a one-parameter family like Cauchy's. Pattern to notice: additive + midpoint = affine. Any functional equation with a midpoint-average form has affine continuous solutions.

Problem 15.8. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the *parallelogram law*

$$f(x+y)+f(x-y) = 2f(x)+2f(y) \qquad \text{for every } x,y \in \mathbb{R}.$$

Technique. Substitutions extract the symmetries of f; induction proves $f(nx) = n^2 f(x)$; extend to $\mathbb{Q}$ and to $\mathbb{R}$.

Discussion. Substitute $x = y = 0$: $2f(0) = 4f(0)$, so $f(0) = 0$. Substitute $x = 0$: $f(y)+f(-y) = 2f(y)$, so $f(-y) = f(y)$: f is even. Substitute $y = x$: $f(2x) = 4f(x)$.

Induction. The pattern $f(nx) = n^2 f(x)$ holds for $n = 0, 1, 2$. For the step, substitute $x \mapsto nx$, $y \mapsto x$:

$$f((n+1)x)+f((n-1)x) = 2f(nx)+2f(x).$$

Assuming the pattern at n and $n-1$,

$$f((n+1)x) = 2n^2 f(x)+2f(x)-(n-1)^2 f(x) = (n+1)^2 f(x).$$

By induction, $f(nx) = n^2 f(x)$ for every positive integer n, and by evenness the same holds for negatives.

Extension. Applying the formula with $x \mapsto x/q$: $f(x) = q^2 f(x/q)$, so $f(x/q) = f(x)/q^2$. Combined with the integer result, $f(rx) = r^2 f(x)$ for every rational r. By continuity and density of $\mathbb{Q}$, the same extends to every real r. Setting $x = 1$ and $c = f(1)$,

$$f(x) = cx^2. \qquad \square$$

What to take away. The parallelogram law characterises the squaring function (up to a multiplicative constant). Its geometric interpretation: in a Euclidean inner-product space, $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$ relates the diagonals of a parallelogram to its sides. Restricted to one dimension, the identity is the functional equation above, and its solutions are the squares cx^2.

OLYMPIAD FINISHERS

PROBLEM 15.9. Find all continuous functions $f : (-1, \infty) \to (-1, \infty)$ satisfying

$$f(xy + x + y) = f(x)f(y) + f(x) + f(y) \qquad \text{for every } x, y > -1.$$

Technique. Substitute $g(x) = f(x) + 1$ to absorb the extra $f(x) + f(y)$ terms; recognise the remaining identity as multiplicative Cauchy after the further substitution $u = x + 1$.

Discussion. Define $g(x) = f(x) + 1$ on $(-1, \infty)$. Since f takes values in $(-1, \infty)$, g takes values in $(0, \infty)$. Writing $f = g - 1$ and expanding the right-hand side of the given identity,

$$f(x)f(y) + f(x) + f(y) = g(x)g(y) - 1.$$

Since $f(xy + x + y) = g(xy + x + y) - 1$, the equation becomes

$$g(xy + x + y) = g(x)g(y).$$

Now note the factorisation $xy + x + y = (x + 1)(y + 1) - 1$. Substitute $u = x + 1 > 0$, $v = y + 1 > 0$, and define $h : (0, \infty) \to$

$(0, \infty)$ by $h(u) = g(u-1)$. The equation becomes

$$h(uv) \;=\; h(u)h(v) \qquad \text{for every } u, v > 0.$$

Taking log on both sides (justified since $h > 0$), $\log h$ is continuous and satisfies $\log h(uv) = \log h(u) + \log h(v)$. By Problem 15.5, $\log h(u) = c \log u$ for some real constant c, hence $h(u) = u^c$. Unwinding,

$$g(x) \;=\; h(x+1) \;=\; (x+1)^c, \qquad f(x) \;=\; (x+1)^c - 1. \qquad \square$$

What to take away. Two substitutions cascade: first, $g = f + 1$ reveals that the right-hand side is $g(x)g(y) - 1$; then $u = x + 1$ reveals that $xy + x + y + 1$ factors as $(x+1)(y+1)$. The problem collapses to multiplicative Cauchy. Chains of substitutions are the standard strategy for olympiad functional equations: the right substitution turns a baroque equation into a textbook one.

PROBLEM 15.10. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x^2 + f(y)) \;=\; y + f(x)^2 \qquad \text{for every } x, y \in \mathbb{R}.$$

Technique. Substitutions to establish bijectivity of f; a symmetry argument forces $f(0) = 0$ and f odd; the equation reduces to additive Cauchy; monotonicity then pins $f(x) = x$.

Discussion. Write $c = f(0)$.

Step 1: f is bijective. Setting $x = 0$ in the given identity,

$$f(f(y)) \;=\; y + c^2.$$

The right-hand side ranges over all of $\mathbb{R}$, so f is surjective. If $f(a) = f(b)$, then $f(f(a)) = f(f(b))$, giving $a + c^2 = b + c^2$ and hence $a = b$: f is injective. So f is a bijection.

Step 2: $f(-x)^2 = f(x)^2$. Replacing x by $-x$ in the given identity leaves the left-hand side unchanged (since $(-x)^2 = x^2$), so

$$y + f(x)^2 \;=\; y + f(-x)^2, \qquad \text{hence} \qquad f(-x)^2 \;=\; f(x)^2.$$

Step 3: f is odd, and $c = 0$. By Step 2, $f(-x) = f(x)$ or $f(-x) = -f(x)$. If $f(-x) = f(x)$ for some $x \neq 0$, injectivity would force

$-x = x$, a contradiction. So $f(-x) = -f(x)$ whenever $x \neq 0$. Separately, $f(1) \neq 0$: the unique y_0 with $f(y_0) = 0$ satisfies $c = f(0) = f(f(y_0)) = y_0 + c^2$, so $y_0 = c - c^2$; if $f(1) = 0$ then $y_0 = 1$, giving $c - c^2 = 1$, which has no real root.

Apply Step 1 at $y = 1$ and at $y = -1$: $f(f(1)) = 1 + c^2$ and $f(f(-1)) = -1 + c^2$. Using $f(-1) = -f(1)$ (from $-1 \neq 0$), and $f(-z) = -f(z)$ at $z = f(1) \neq 0$,

$$f(f(-1)) = f(-f(1)) = -f(f(1)) = -(1 + c^2).$$

Equating the two expressions for $f(f(-1))$:

$$-1 + c^2 = -1 - c^2, \qquad 2c^2 = 0, \qquad c = 0.$$

So $f(0) = 0$, and f is odd on all of $\mathbb{R}$.

Step 4: Additivity. With $c = 0$, Step 1 becomes $f(f(y)) = y$: f is an involution. Setting $y = 0$ in the given identity,

$$f(x^2) = f(x)^2 \geq 0.$$

So f maps $[0, \infty)$ into $[0, \infty)$.

Setting $y \mapsto f(w)$ in the given identity (using $f(f(w)) = w$), for every $a = x^2 \geq 0$ and every w,

$$f(a + w) = f(a) + f(w).$$

Extending to negative a: if $a < 0$, use $f(a + w) = -f(-a - w) = -(f(-a) + f(-w)) = f(a) + f(w)$ (applying the $[0, \infty)$ case to $-a > 0, -w$, then using oddness). So $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$.

Step 5: Monotonicity and conclusion. f is additive and non-negative on $[0, \infty)$. For $b \geq a, f(b) = f(a) + f(b - a) \geq f(a)$: f is non-decreasing. The argument of Problem 15.3 applies (monotonicity plays the role of continuity in extending $\mathbb{Q}$-linearity to $\mathbb{R}$-linearity), so $f(x) = kx$ for some constant k.

Finally, $f(f(x)) = x$ gives $k^2 x = x$, so $k^2 = 1$. Monotone non-decreasing forces $k \geq 0$, hence $k = 1$. Therefore

$$f(x) = x.$$

A check: $f(x^2 + f(y)) = x^2 + y$ and $y + f(x)^2 = y + x^2$, which agree. □

What to take away. The book's final problem bears the stamp of every technique the reader has learned. Substitution tricks isolate individual values. Injectivity and bijectivity of f crack open the shape of the equation. A key algebraic symmetry ($f(-x)^2 = f(x)^2$, combined with injectivity) forces f to be odd. The endgame reduces to Cauchy's additive equation and its monotone-plus-additive corollary. The unique answer, $f(x) = x$, is satisfying in its starkness: out of every function on $\mathbb{R}$, only the identity satisfies the equation, and the seven-step proof is assembled from ordinary algebraic moves, none of them unfamiliar. This is what competition mathematics, at its finest, is about: a succession of small tricks, each elementary, whose composition cracks an apparently intractable problem.