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How Far Can You Roll?

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A puzzle from Fiddler on the Proof, in which every path length is an elliptic integral and the average of them is a fraction.

Problem

Frederica Fiddleria attaches a light to a point on the circumference of a circular wheel with a radius of one metre. She rolls the wheel for one revolution along the ground, and the light traces a path. What is the length of that path?

One idea does the whole puzzle

A rolling wheel does not slip, and that single fact fixes the speed of every point on it. The point touching the ground is momentarily stationary, so at each instant the wheel is turning about that contact point, exactly as though it were hinged there. Every point of the wheel is therefore moving in a circle about the contact point, and its speed is

$$\text{speed} \;=\; \omega \times (\text{distance to the contact point}).$$

Measure time by the turning angle t, so that $\omega = 1$ and one revolution is t running from 0 to 2π. Then the speed of a point is simply its distance from the point of contact, and the length of its path is

$$L \;=\; \int_0^{2\pi} (\text{distance to the contact point})\, \mathrm{d}t.$$

That is the whole method. Both the puzzle and its extra credit are now questions about a distance in a circle rather than questions about motion.

The light on the rim

Put the light on the rim. The contact point is also on the rim, and the angle between them, measured at the centre, is exactly the angle t the wheel has turned through. Two points on a circle of radius 1 separated by a central angle t are a chord apart, and that chord has length $2\sin(t/2)$. So the speed of the light is $2\sin(t/2)$, which is zero at the start, when the light is on the ground, and 2 at the top, twice the speed of the centre.

$$L \;=\; \int_0^{2\pi} 2\sin\tfrac{t}{2}\,\mathrm{d}t \;=\; \left[-4\cos\tfrac{t}{2}\right]_0^{2\pi} \;=\; 4+4.$$

$$\boxed{L = 8 \text{ metres.}}$$

The curve is a cycloid, and the result is the classical one: an arch of a cycloid has length eight times the radius, a fact that predates calculus. It is worth a moment that the answer is a whole number and that π has vanished from a question about a circle. The wheel travels 2π along the ground while the light travels 8, so the light covers about 27 per cent more distance than the wheel does.

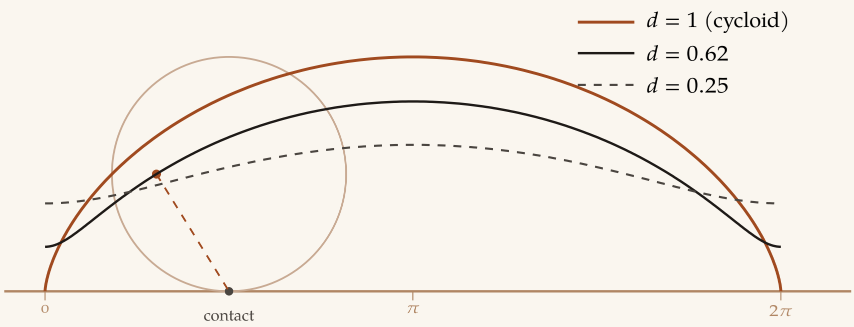

Figure 1: *One revolution, and the path traced by a point at distance d from the centre. The rim point* ($d = 1$) *traces a cycloid with a cusp at the ground, where it is momentarily at rest. Points inside the wheel trace curtate cycloids, which never stop and never reach the ground. The dashed line is the distance to the contact point, which is the point's speed.*

Extra Credit

Instead of placing the light on the circumference, Frederica picks a random point inside the circle, with any two regions of the same area equally likely to contain it. On average, what can she expect the length of the path to be?

One path at a time

Take a point at distance d from the centre, where $0 \le d \le 1$. When the wheel has turned through t, the angle at the centre between our point and the contact point is t, and the two are at distances d and 1 from the centre. The law of cosines gives the distance between them, which is the speed:

$$\text{speed} \;=\; \sqrt{1 + d^2 - 2d\cos t}.$$

At $d = 1$ this is $\sqrt{2 - 2\cos t} = 2\sin(t/2)$, agreeing with the rim case. The path length for that point is

$$L(d) \;=\; \int_0^{2\pi} \sqrt{1 + d^2 - 2d\cos t}\,\mathrm{d}t,$$

and this integral has no elementary form. Writing $\cos t = 1 - 2\sin^2(t/2)$ turns it into a complete elliptic integral of the second kind,

$$L(d) \;=\; 4\,(1+d)\,E(k), \qquad k = \frac{2\sqrt{d}}{1+d},$$

which checks out at both ends: $L(1) = 8E(1) = 8$, and $L(0) = 4E(0) = 2\pi$, the centre travelling in a straight line one circumference long. So the answer to a single point is not a nice number, and there is no reason yet to expect the average of such things to be nice.

All the paths at once

Do not average the elliptic integrals. Average the distances instead.

A uniformly random point in the disc has radius density $2d$ on $[0,1]$, so

$$\mathbb{E}[L] \;=\; \int_0^1 2d\,L(d)\,\mathrm{d}d \;=\; \int_0^1\!\!\int_0^{2\pi} 2d\,r(d,t)\,\mathrm{d}t\,\mathrm{d}d,$$

where $r(d,t)$ is the distance from the point at polar coordinates (d,t) to the contact point. The element of area in polar coordinates is $\mathrm{d}A = d\,\mathrm{d}t\,\mathrm{d}d$, so that double integral is an area integral over the whole disc, and

$$\mathbb{E}[L] \;=\; 2\iint_{\text{disc}} r\,\mathrm{d}A.$$

The two variables have swapped roles. What began as an average over points of an integral over time has become a single integral over the disc, because the angle t that describes the rolling is the same angle that sweeps out the disc. The contact point sits on the rim, so all that is left is the integral of distance from a fixed point of the circumference.

That is an easy integral in the right coordinates. Put the origin at the rim point. The unit disc is then described by $r \le 2\cos\varphi$ for $-\pi/2 \le \varphi \le \pi/2$, since a chord from a point on a circle of radius 1 makes that length at angle φ to the diameter. So

$$\iint_{\text{disc}} r\,\mathrm{d}A = \int_{-\pi/2}^{\pi/2}\!\!\int_0^{2\cos\varphi} r\cdot r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{-\pi/2}^{\pi/2} \frac{8\cos^3\varphi}{3}\,\mathrm{d}\varphi = \frac{8}{3}\cdot\frac{4}{3} = \frac{32}{9},$$

using $\int_{-\pi/2}^{\pi/2}\cos^3\varphi\,\mathrm{d}\varphi = 4/3$. Hence

$$\boxed{\mathbb{E}[L] \;=\; \frac{64}{9} \;=\; 7.1111\ldots\ \text{metres.}}$$

Every individual path length is an elliptic integral, and not one of them is rational except the two endpoints. The average is 64/9. The π in the area element cancels against the π hiding in the mean distance, which is $32/(9\pi)$, and a transcendental family averages to a fraction.

The number sits where it should: between $2\pi = 6.28$ for the centre and 8 for the rim, and nearer the top of that range because area weighting favours points far from the centre.

The computation

The check should roll the wheel rather than evaluate either formula. Take a uniform point in the disc, trace it through one revolution, and measure the polyline.

```
import numpy as np
from math import pi

rng = np.random.default_rng(7)
M, NT = 400_000, 3000
t = np.linspace(0.0, 2*pi, NT + 1)
d = np.sqrt(rng.random(M))        # uniform in the disc: radius density 2d

lens = np.empty(M)
i = 0
for chunk in np.array_split(d, 80):
    x = t[None, :] - chunk[:, None]*np.sin(t)[None, :] # roll it
    y = 1.0 - chunk[:, None]*np.cos(t)[None, :]
    lens[i:i+len(chunk)] = np.hypot(np.diff(x, axis=1),
                                    np.diff(y, axis=1)).sum(axis=1)
    i += len(chunk)

print(f"{lens.mean():.6f} +- {lens.std()/np.sqrt(M):.6f}")
# 7.111089 +- 0.000779    64/9 = 7.111111
```

Four hundred thousand traced paths give 7.111089 ± 0.000779, which is three hundredths of a standard error from $64/9$. Setting $d = 1$ in the same tracer returns 8.00000000 for the main puzzle.

Two further checks, both independent of the derivation. Gauss-Legendre quadrature of $L(d)$ against the elliptic form $4(1+d)E(k)$ agrees to 1.3×10^{-13} across d from 0 to 1. And sampling points in the disc directly, the mean distance to a rim point comes to 1.1321 against $32/(9\pi) = 1.13177$, which multiplied by 2π returns $64/9$ again by the geometric route rather than the kinematic one.