

Can You Frame the Triangle?



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A puzzle from Fiddler on the Proof, in which a random spin averages out to a fraction with a π in it.

Problem

A paper football is an equilateral triangle of side one inch. What is the side of the smallest square frame that will hold it?

Solution

Put the triangle's centre at the origin. A unit equilateral triangle has circumradius $1/\sqrt{3}$, so after turning through an angle θ its vertices sit at angles θ , $\theta + 120^\circ$ and $\theta + 240^\circ$ at that distance from the centre.

The frame is a square, so it must be as wide as the triangle and as tall, and its side is the larger of the two. Both extents are worth writing down. The horizontal one is the spread of the three vertex x -coordinates, and for θ between 0 and 60° the widest pair are the vertices at θ and $\theta + 120^\circ$:

$$w(\theta) = \frac{1}{\sqrt{3}}(\cos \theta - \cos(\theta + 120^\circ)) = \sin(\theta + 60^\circ),$$

using $\cos A - \cos B = 2 \sin \frac{A+B}{2} \sin \frac{B-A}{2}$. The vertical extent is the same function measured a quarter turn round, so $h(\theta) = w(\theta - 90^\circ)$, which on $0 \leq \theta \leq 30^\circ$ comes to $\cos \theta$.

So on that range the frame has side $\max(\sin(\theta + 60^\circ), \cos \theta)$. The first term climbs and the second falls, so the largest of the two is smallest where they cross:

$$\sin(\theta + 60^\circ) = \cos \theta = \sin(90^\circ - \theta) \implies \theta = 15^\circ,$$

and there the frame is square by construction, with side $\sin 75^\circ$. At that angle the triangle touches all four sides of the frame, with the vertex that is both leftmost and highest sitting exactly in a corner.

$$\text{side} = \cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.9659 \text{ inches.}$$

Left alone, with one side of the triangle horizontal, the frame would need a full inch, since the triangle's own side is an inch and its height is only $\sqrt{3}/2$. Tilting by 15° buys about 3.4 per cent off the side of the frame, which is the puzzle's small surprise: the tidy orientation is not the economical one.

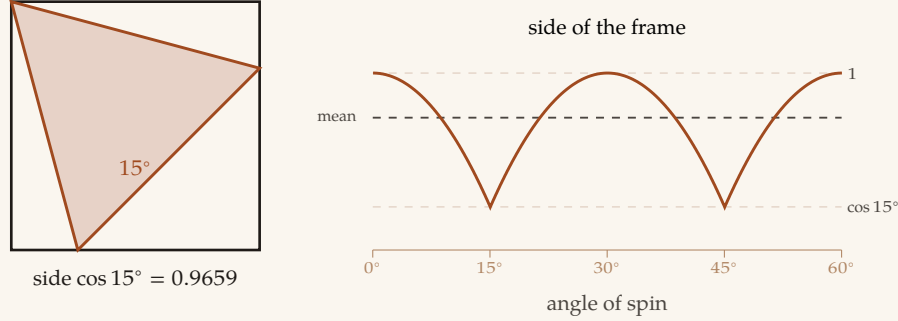


Figure 1: Left: the triangle turned 15° , touching all four sides of its smallest frame. Right: the side of the smallest frame as the triangle spins, over one full period of 60° . The curve is four arcs of a sine, each running between $\cos 15^\circ$ and 1.

Extra Credit

Now spin the football through a uniformly random angle and frame it with a square whose sides are horizontal and vertical. On average, what side does the frame need?

Four arcs of one sine curve

The answer is an average of $s(\theta) = \max(w(\theta), h(\theta))$ over a uniform angle, and the work is in seeing how little of the circle needs looking at.

The triangle carries itself onto itself under a third of a turn, so w repeats every 120° . Swapping the two extents is a quarter turn, and together these make s repeat every 60° . Inside one such period the two extents trade places at 15° , 30° and 45° , cutting the period into four arcs of 15° each, and on every arc s is a piece of a sine curve with its argument running between 75° and 90° in one direction or the other.

That is the whole calculation, because a sine is symmetric about its peak. Each of the four arcs therefore contributes the same amount,

$$\int_{75^\circ}^{90^\circ} \sin x \, dx = \cos 75^\circ,$$

and dividing the total by the length of the period gives

$$\mathbb{E}[\text{side}] = \frac{4 \cos 75^\circ}{\pi/3} = \frac{12 \cos 75^\circ}{\pi} = \frac{3(\sqrt{6} - \sqrt{2})}{\pi} \approx 0.9886 \text{ inches.}$$

Two things are worth noticing about that number. It lies between the best frame, $\cos 15^\circ = 0.9659$, and the worst, which is exactly 1 and happens whenever a side of the triangle is horizontal or vertical. And it lies above the midpoint of those two, 0.9830, because a sine curve flattens near its peak: a spinning triangle spends longer close to its widest than close to its narrowest.

A random spin therefore costs about two and a third per cent more frame than careful placement, and still beats the tidy orientation, which is the worst case.

The computation

The check works from the three vertex coordinates and assumes none of the algebra above: it rotates the triangle, takes the spread of the coordinates in each direction, and keeps the larger.

```
import numpy as np
from math import pi, sqrt

R = 1/sqrt(3)                                # circumradius of a unit equilateral triangle

def side(theta):                              # side of the smallest axis-aligned square
    a = theta + np.array([0.0, 2*pi/3, 4*pi/3])
    x, y = R*np.cos(a), R*np.sin(a)
    return max(x.max() - x.min(), y.max() - y.min())
```

Scanning 600,001 rotations puts the smallest frame at 0.965925826, reached at 15.0000° , against $\cos 15^\circ = 0.965925826$, and the largest at exactly 1. Integrating $s(\theta)$ over one 60° period by Simpson's rule gives 0.988615929 against $3(\sqrt{6} - \sqrt{2})/\pi = 0.988615929$, agreeing to 3×10^{-16} .

One check earns its place. The claim that the period is 60° and not something longer is easy to assume and easy to get wrong, so the same average was taken over a full turn instead: it returns 0.988615986, the difference being the coarser grid rather than a different answer.